P2

(a) Let $F(t') = \frac{2}{\pi}\int_0^\infty [\epsilon_1(\omega')-1]\cos(\omega' t')d\omega' = \frac{2}{\pi}\int_0^\infty \epsilon_2(\omega')\sin(\omega' t')\,d\omega'$

Now, in the notes we defined:

$$\epsilon_1(\omega)-1 = \int_0^\infty F(t')\cos(\omega t')dt'$$

$$= \int_0^\infty \cos(\omega t')dt' \left\{\frac{2}{\pi}\int_0^\infty \epsilon_2(\omega')\sin(\omega' t')\,d\omega'\right\}$$

$$= \frac{2}{\pi}\int_0^\infty d\omega' \epsilon_2(\omega') \left\{\int_0^\infty \cos(\omega t')\sin(\omega' t')\,dt'\right\}$$

Now, $\cos(\omega t') = \frac{e^{i\omega t'}+e^{-i\omega t'}}{2}$ and $\sin(\omega' t') = \frac{e^{i\omega' t'}-e^{-i\omega' t'}}{2i}$ $\therefore$

$$\epsilon_1(\omega)-1 = \frac{2}{\pi}\int_0^\infty d\omega' \epsilon_2(\omega')\left\{\int_0^\infty \left(\frac{e^{it'(\omega+\omega')}-e^{it'(\omega-\omega')}+e^{it'(\omega'-\omega)}-e^{-it'(\omega+\omega')}}{4i}\right)\right.$$

Now, the Laplace transform is defined as:

$$\mathcal{L}\{f(t)\} = \int_0^\infty e^{-\alpha t} f(t)dt \quad \therefore$$

$$\int_0^\infty e^{-\alpha t'} e^{-\beta t'}dt' = \frac{1}{\alpha+\beta}.$$

With this definition, we have:

$$A = \frac{1}{4i}\left(e^{it'(\omega+\omega')} - e^{it'(\omega-\omega')} + e^{it'(\omega'-\omega)} - e^{-it'(\omega+\omega')}\right)$$

$$= \frac{1}{4i}\left(\frac{1}{-i\omega'-i\omega} - \frac{1}{+i\omega'-i\omega} + \frac{1}{-i\omega'+i\omega} - \frac{1}{i\omega'+i\omega}\right);$$

So, multiplying through by $i/i$ gives:

$$A = \frac{1}{4i}\left( \frac{i}{\omega'+\omega} + \frac{i}{\omega'-\omega} - \frac{i}{\omega-\omega'} + \frac{i}{\omega'+\omega} \right)$$

$$= \frac{1}{4i}\left( \frac{2i}{\omega'+\omega} + \frac{2i}{\omega'-\omega} \right) = \frac{1}{4i}\left( \frac{2i(\omega'-\omega) + 2i(\omega'+\omega)}{\omega'^2-\omega^2} \right)$$

$$= \frac{1}{4i}\left( \frac{2i\omega' - 2i\omega + 2i\omega' + 2i\omega}{\omega'^2-\omega^2} \right) = \frac{1}{4i}\left( \frac{4i\omega'}{\omega'^2-\omega^2} \right)$$

Thus,

$$\boxed{\epsilon_1(\omega) - 1 = \frac{2}{\pi} P\int_0^\infty d\omega'\, \epsilon_2(\omega')\, \frac{\omega'}{\omega'^2-\omega^2}}$$

Similarly, for $\epsilon_2(\omega)$:

$$\epsilon_2(\omega) = \int_0^\infty F(t')\sin(\omega t')dt'$$

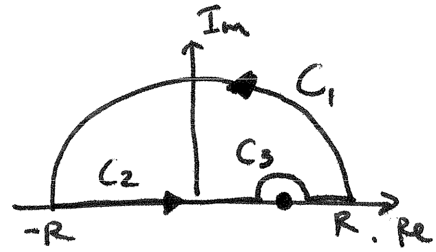
$$= \int_0^\infty \sin(\omega t')dt' \left\{ \frac{2}{\pi}\int_0^\infty [\epsilon_1(\omega')-1]\cos(\omega' t')d\omega' \right\}$$

$$= \frac{2}{\pi}\int_0^\infty d\omega' [\epsilon_1(\omega')-1] \left\{ \int_0^\infty \sin(\omega t')\cos(\omega t')dt' \right\}$$

$$= \frac{2}{\pi}\int_0^\infty d\omega' [\epsilon_1(\omega')-1] \left\{ (A') \right\}$$, where $A'$ has $\omega$ and $\omega'$ switched and multiplied by a minus sign.

$$\boxed{\epsilon_2(\omega) = -\frac{2}{\pi}\int_0^\infty d\omega' [\epsilon_1(\omega')-1] \frac{\omega}{\omega'^2-\omega^2}}$$

(b)




$$\frac{\epsilon(\omega)}{\epsilon_0} = 1 + \frac{1}{2\pi i} \oint_C \frac{\left(\frac{\epsilon(\omega')}{\epsilon_0} - 1\right)}{\omega' - \omega} d\omega'$$

$$\rightarrow \oint_C = \oint_{C_1} + \oint_{C_2} + \oint_{C_3} = 0$$

Now, if we let $R \to \infty$, $\oint_{C_1} \to 0$. Then:

$$\oint_C = P\int_{-\infty}^{\infty} \frac{\frac{\epsilon(\omega')}{\epsilon_0} - 1}{\omega' - \omega} d\omega' + i\pi \left(\frac{\epsilon(\omega)}{\epsilon_0} - 1\right) = 0$$

With this, we have that:

$$\frac{\epsilon(\omega)}{\epsilon_0} = 1 - \frac{1}{i\pi} P\int_{-\infty}^{\infty} \frac{\frac{\epsilon(\omega')}{\epsilon_0} - 1}{\omega' - \omega} d\omega'$$

Now, $\epsilon(\omega) = \epsilon_1 + i\epsilon_2$ $\therefore$

$$\frac{\epsilon_1}{\epsilon_0} + \frac{i\epsilon_2}{\epsilon_0} = 1 - \frac{1}{i\pi} P\int_{-\infty}^{\infty} \frac{\frac{\epsilon_1}{\epsilon_0} + \frac{i\epsilon_2}{\epsilon_0} - 1}{\omega' - \omega} d\omega' \quad \therefore$$

$\frac{\epsilon_1}{\epsilon_0}$ corresponds to real parts of RHS and

$\frac{\epsilon_2}{\epsilon_0}$ corresponds to imaginary $\therefore$

$$\frac{\epsilon_1(\omega)}{\epsilon_0} = 1 + \frac{1}{\pi} P\int_{-\infty}^{\infty} \frac{\frac{\epsilon_2(\omega')}{\epsilon_0}}{\omega' - \omega} d\omega'$$

$$\frac{\epsilon_2(\omega)}{\epsilon_0} = -\frac{1}{\pi} P\int_{-\infty}^{\infty} \frac{\frac{\epsilon_1(\omega')}{\epsilon_0} - 1}{\omega' - \omega} d\omega'$$

$$\frac{d}{dw}\left(\frac{1}{w'^2 - w^2}\right)$$

$$u = w'^2 - w^2$$

$$\frac{du}{dw} = -2w$$

$$= \frac{1}{u}$$

$$\frac{df}{dw} = \frac{df}{du}\,\frac{du}{dw}.$$

$$\frac{d}{du}(u^{-1}) = \frac{-1}{u^2}.$$

$$\frac{-1}{u^2} \cdot (-2w) = \frac{2w}{(w'^2 - w^2)^2}$$

$$\frac{1}{w'^2 - w^2} \;?$$

P3

$\vec{S} = c\vec{E}\times\vec{H}/4\pi$

$$-\vec{\nabla}\cdot\vec{S} = \frac{1}{4\pi}\left(\vec{E}\cdot\frac{\partial\vec{D}}{\partial t} + \vec{H}\cdot\frac{\partial\vec{B}}{\partial t}\right)$$

Now, assume $\vec{E} \to \frac{1}{2}\vec{E}' e^{-i\omega t} + \frac{1}{2}\vec{E}'^{*} e^{i\omega t}$

$\vec{H} \to \frac{1}{2}\vec{H}' e^{-i\omega t} + \frac{1}{2}\vec{H}'^{*} e^{i\omega t}$

Then,

$\vec{D} = \frac{1}{2}\epsilon\vec{E}' e^{-i\omega t} + \frac{1}{2}\epsilon^{*}\vec{E}'^{*} e^{i\omega t}$ , $\vec{E} = \frac{1}{2}\vec{E}' e^{-i\omega t}$

$\frac{\partial\vec{D}}{\partial t} = \frac{1}{2}\left(-i\omega\epsilon\vec{E} + i\omega\epsilon^{*}\vec{E}^{*}\right)$ $\quad \vec{H} = \frac{1}{2}\vec{H}' e^{-i\omega t}$

$\vec{B} = \frac{1}{2}\mu\vec{H}' e^{-i\omega t} + \frac{1}{2}\mu^{*}\vec{H}' e^{i\omega t}$

$\frac{\partial\vec{B}}{\partial t} = \frac{1}{2}\left(-i\omega\mu\vec{H} + i\omega\mu^{*}\vec{H}^{*}\right)$

$$-\vec{\nabla}\cdot\vec{S} = \frac{1}{4\pi}\Big(\frac{1}{4}\left(-i\omega\epsilon\,\vec{E}\cdot\vec{E} + i\omega\epsilon^{*}\vec{E}\cdot\vec{E}^{*} - i\omega\epsilon\vec{E}^{*}\cdot\vec{E} + i\omega\epsilon^{*}\vec{E}^{*}\cdot\vec{E}^{*}\right)$$

$$+\frac{1}{4}\left(-i\omega\mu\vec{H}\cdot\vec{H} + i\omega\mu^{*}\vec{H}\cdot\vec{H}^{*} - i\omega\mu\vec{H}^{*}\cdot\vec{H} + i\omega\mu^{*}\vec{H}^{*}\cdot\vec{H}^{*}\right)\Big)$$

$$= \frac{i\omega}{16\pi}\left((\epsilon^{*}-\epsilon)\vec{E}\cdot\vec{E}^{*} + (\mu^{*}-\mu)\vec{H}\cdot\vec{H}^{*}\right) + \frac{i\omega}{16\pi}\Big(\epsilon\vec{E}\cdot\vec{E} + \epsilon^{*}\vec{E}^{*}\cdot\vec{E}^{*}$$

$$+\mu\vec{H}\cdot\vec{H} + \mu^{*}\vec{H}^{*}\cdot\vec{H}^{*}\Big)$$

Taking a time average of the $-\vec{\nabla}\cdot\vec{S}$ function over one cycle gives the heat dissipation. This will eliminate the $\vec{E}^*\cdot\vec{E}^*$ and $\vec{H}^*\cdot\vec{H}^*$ terms because the average over the cycle is zero. Thus,

$$Q = \frac{i\omega}{16\pi}\left[(\epsilon^* - \epsilon)\vec{E}\cdot\vec{E}^* + (\mu^* - \mu)\vec{H}\cdot\vec{H}^*\right] \therefore$$

$$Q = \frac{\omega}{8\pi}\left[\epsilon_2|\vec{E}|^2 + \mu_2|\vec{H}|^2\right] = \frac{\omega}{4\pi}\left[\epsilon_2\vec{E}^2 + \mu_2\vec{H}^2\right]$$

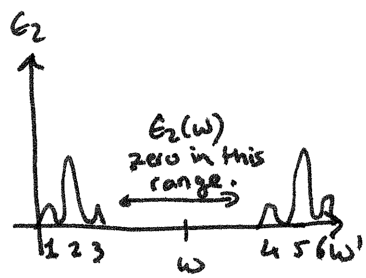
Now, by entropic considerations, the dissipation of energy results in the positive (always) generation of heat. Thus, given the equation for $Q$ above, the only way to ensure $Q>0$ is for $\epsilon_2>0$ and $\mu_2>0$.

P4

(a) While $\epsilon_2(\omega)$ may equal 0 at some given frequency $\omega$, it is not necessarily zero at all frequencies. If it were, according to the Kramers-Kronig relation:

$$\epsilon_1(\omega) = 1 + \frac{2}{\pi} P \int_0^\infty \frac{\epsilon_2(\omega')\,\omega'}{\omega'^2 - \omega^2}\, d\omega'$$

$\epsilon_1(\omega)$ would have to be 1 and $d\epsilon_1/d\omega = 0$.




So, in general, $\epsilon_2(\omega)$ will be zero at one or more frequencies, but may have peaks at others. For this problem, we assume that all the peaks are far away from $\omega$. Then:

$$\epsilon_1(\omega) = 1 + \frac{2}{\pi}\left[\int_{\text{peak 1}} \frac{\epsilon_2(\omega')\omega'}{\omega'^2-\omega^2}\, d\omega' + \int_{\text{peak 2}} \frac{\epsilon_2(\omega')\omega'}{\omega'^2-\omega^2}\, d\omega' + \cdots\right]$$

$$+ \frac{2}{\pi}\left[\int_{\text{peak 3}} \frac{\epsilon_2(\omega')\omega'}{\omega'^2-\omega^2}\, d\omega' + \cdots\right]$$

The second term on the RHS accounts for all peaks before $\omega$. There, we can take $\omega^2 \gg \omega'^2$ so $(\omega'^2 - \omega^2) \approx -\omega^2$.

The third term on the RHS accounts for all peaks after $\omega$. There, we can take $\omega'^2 \gg \omega^2$ so $(\omega'^2 - \omega^2) \approx \omega'^2$.

With these simplifications, the expression for $\epsilon_1(\omega)$ becomes:

$$\epsilon_1(\omega) \approx 1 - \frac{2}{\pi\omega^2}\left[\int_{\text{peak 1}} \epsilon_2(\omega')\omega' d\omega' + \cdots\right] \qquad \text{—— (A)}$$

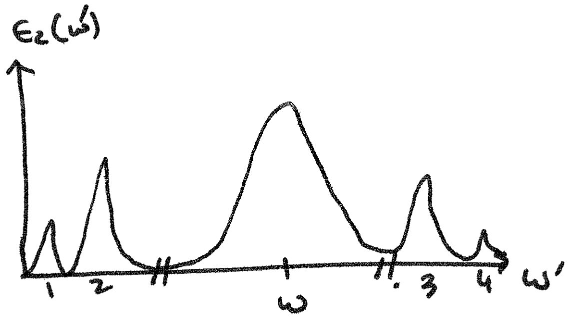
$$+ \frac{2}{\pi}\left[\int_{\text{peak 3}} \frac{\epsilon_2(\omega')\omega'}{\omega'^2} d\omega' + \cdots\right]$$

Then, $d\epsilon_1/d\omega$ is:

$$\boxed{\frac{d\epsilon_1}{d\omega} = + \frac{4}{\pi\omega^3}\left[\int_{\text{peak 1}} \epsilon_2(\omega')\omega' d\omega' + \cdots\right] > 0}$$

↳ for positive $\omega$

(b) Now, if $\epsilon_2(\omega) \neq 0$, we have an absorption that looks like:




Now, we must add a 4th term onto the RHS of equation (A) to account for the peak around $\omega$. Thus:

$$\epsilon_1(\omega) \approx 1 - \frac{2}{\pi\omega^2}\left[\int_{P1} \epsilon_2(\omega')\omega' d\omega' + \cdots\right]$$

$$+ \frac{2}{\pi}\left[\int_{P3} \frac{\epsilon_2(\omega')\omega'}{\omega'^2} d\omega' + \cdots\right]$$

$$+ \frac{2}{\pi}\int_{\substack{\text{peak} \\ \text{around } \omega}} \frac{\epsilon_2(\omega')\omega'}{(\omega'^2 - \omega^2)} d\omega'$$

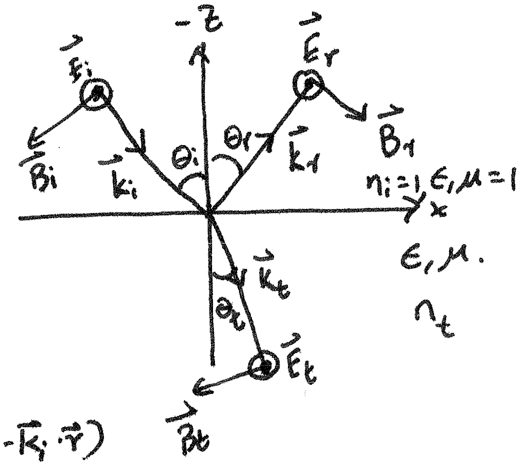
Now, $d\epsilon_1/d\omega$ is:

$$\frac{d\epsilon_1}{d\omega} = \frac{4}{\pi\omega^3}\left[\int_{P1} \epsilon_2(\omega')\omega' d\omega' + \cdots\right] + \frac{8\omega}{\pi}\int \frac{\epsilon_2(\omega')\,\omega'}{(\omega'^2-\omega^2)^2}\,d\omega'$$

peak around $\omega$

Since the second term can be negative, it is possible that $\frac{d\epsilon_1}{d\omega} < 0$ because the integral $\int \frac{\epsilon_2(\omega')\omega'}{(\omega'^2-\omega^2)^2}d\omega'$ can be negative (e.g. $\epsilon_2(\omega') = 1$).

For simplicity, we consider s-polarization. Then,




Incident Beam

$$\vec{E}_i = \hat{j} E_i e^{i(\omega t - \vec{k}_i \cdot \vec{r})}$$

$$\vec{B}_i = \frac{1}{\omega} \vec{k}_i \times \vec{E}_i$$

$$\vec{k}_i = \frac{\omega n_i}{c} (\sin\theta_i \hat{\imath} + \cos\theta_i \hat{n}) \quad \therefore$$

$$\vec{B}_i = \frac{n_i}{c} (-\cos\theta_i \hat{\imath} + \sin\theta_i \hat{n}) E_i e^{i(\omega t - \vec{k}_i \cdot \vec{r})}$$

Reflected Beam

$$\vec{E}_r = \hat{j} E_r e^{i(\omega t - \vec{k}_r \cdot \vec{r})} \qquad \vec{k}_r = (\sin\theta_i \hat{\imath} - \cos\theta_i \hat{n}) \frac{\omega n_i}{c}$$

$$\vec{B}_r = \frac{n_i}{c} (\cos\theta_i \hat{\imath} + \sin\theta_i \hat{n}) E_r e^{i(\omega t - \vec{k}_r \cdot \vec{r})}$$

Now, we consider boundary conditions. The normal component of $\vec{D}$ is continuous across the boundary, and this is satisfied because the s-polarization has no normal component. The boundary condition on the tangential component requires that $E_i + E_r = E_t$. Similar conditions for the normal and tangential components of the B-field lead to:

$$\frac{n_i}{c} \sin\theta_i (E_i + E_r) = \frac{n_t}{c} \sin\theta_t E_t$$

$$\frac{n_i}{\mu_i c} \cos\theta_i (E_i - E_r) = \frac{n_t}{\mu_t c} \cos\theta_t E_t$$

Here we have used the fact that the transmitted beam goes like:

$$\vec{E}_t = \hat{j} E_t e^{i(\omega t - \vec{k}_t \cdot \vec{r})} \qquad \vec{k}_t = (\sin\theta_t \hat{\imath} + \cos\theta_t \hat{n}) \frac{\omega n_t}{c}$$

$$\vec{B}_t = (-\cos\theta_t \hat{\imath} + \sin\theta_t \hat{n}) \frac{n_t}{c} E_t e^{i(\omega t - \vec{k}_t \cdot \vec{r})}$$

We can eliminat $E_t$ from the tangential equation using $E_i + E_r = E_t$

$$\frac{n_i}{\mu_i c}\cos\theta_i\,(E_i - E_r) = \frac{n_t}{\mu_t c}\cos\theta_t\,(E_i + E_r) \therefore$$

$$E_i\left[\frac{n_i}{\mu_i}\cos\theta_i - \frac{n_t}{\mu_t}\cos\theta_t\right] = E_r\left[\frac{n_i}{\mu_i}\cos\theta_i + \frac{n_t}{\mu_t}\cos\theta_t\right] \therefore$$

$$\frac{E_r}{E_i} = \frac{\frac{n_i}{\mu_i}\cos\theta_i - \frac{n_t}{\mu_t}\cos\theta_t}{\frac{n_i}{\mu_i}\cos\theta_i + \frac{n_t}{\mu_t}\cos\theta_t} \therefore$$

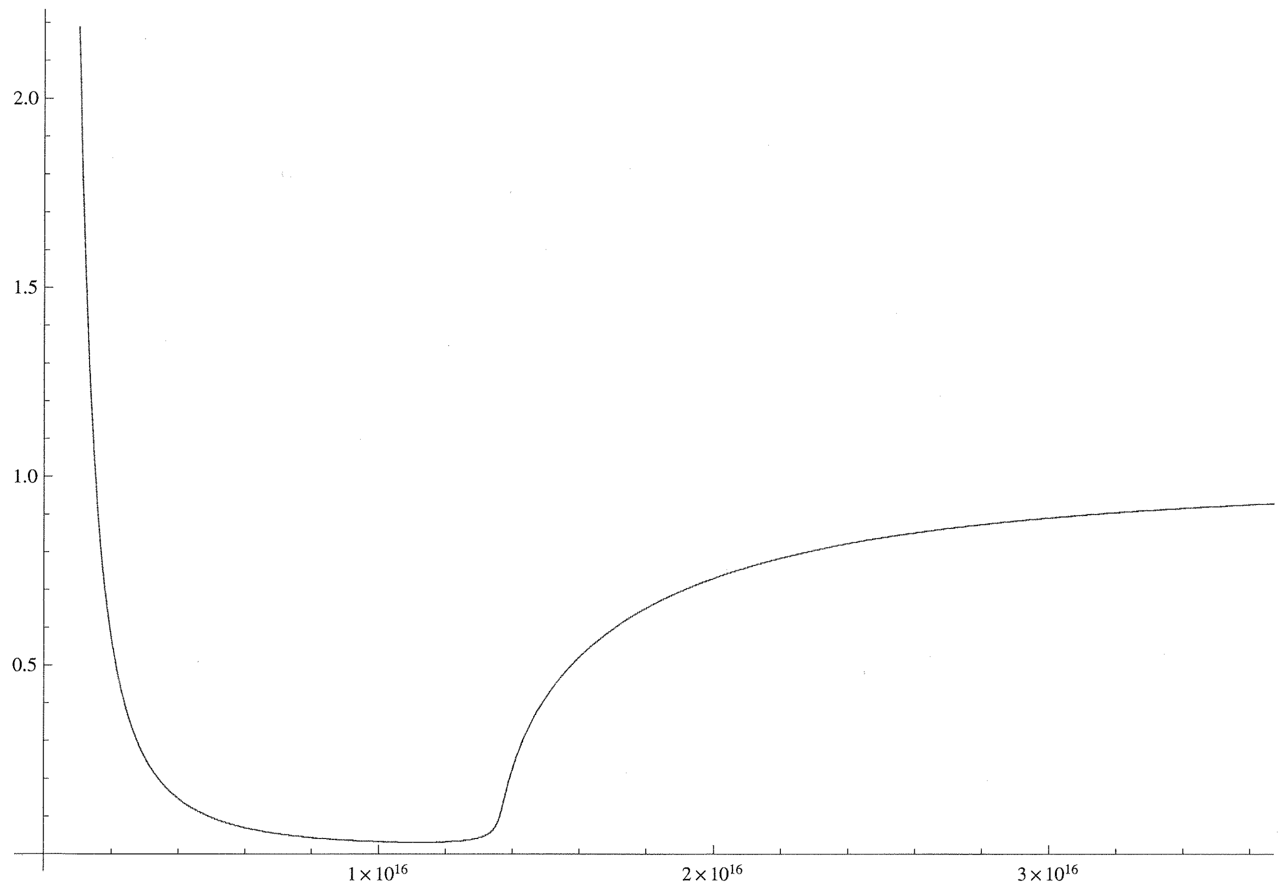
$$\boxed{\frac{E_r}{E_i} = \frac{\sqrt{\frac{\epsilon_i}{\mu_i}}\cos\theta_i - \sqrt{\frac{\epsilon_t}{\mu_t}}\cos\theta_t}{\sqrt{\frac{\epsilon_i}{\mu_i}}\cos\theta_i + \sqrt{\frac{\epsilon_t}{\mu_t}}\cos\theta_t}}$$

For a monochromatic wave impinging on a left handed material slab, we would expect negative refraction according to Snell's law. Also, in order to conserve momentum, the "k" vector in the slab would point towards the initial interface. However, since the poynting vector is $\propto \vec{E} \times \vec{H}$, the energy flow would still be in the incident $\rightarrow$ transmitted direction as expected. This is also true for a wave packet and we would expect negative refraction according to Snell's law at the central frequency and a incident $\rightarrow$ transmitted poynting vector to indicate unidirection energy flow through the slab. However due to the superposed nature of a wave packet, we would expect spatial ~~[illegible]~~ distortion in the packet as it propagates through the slab.

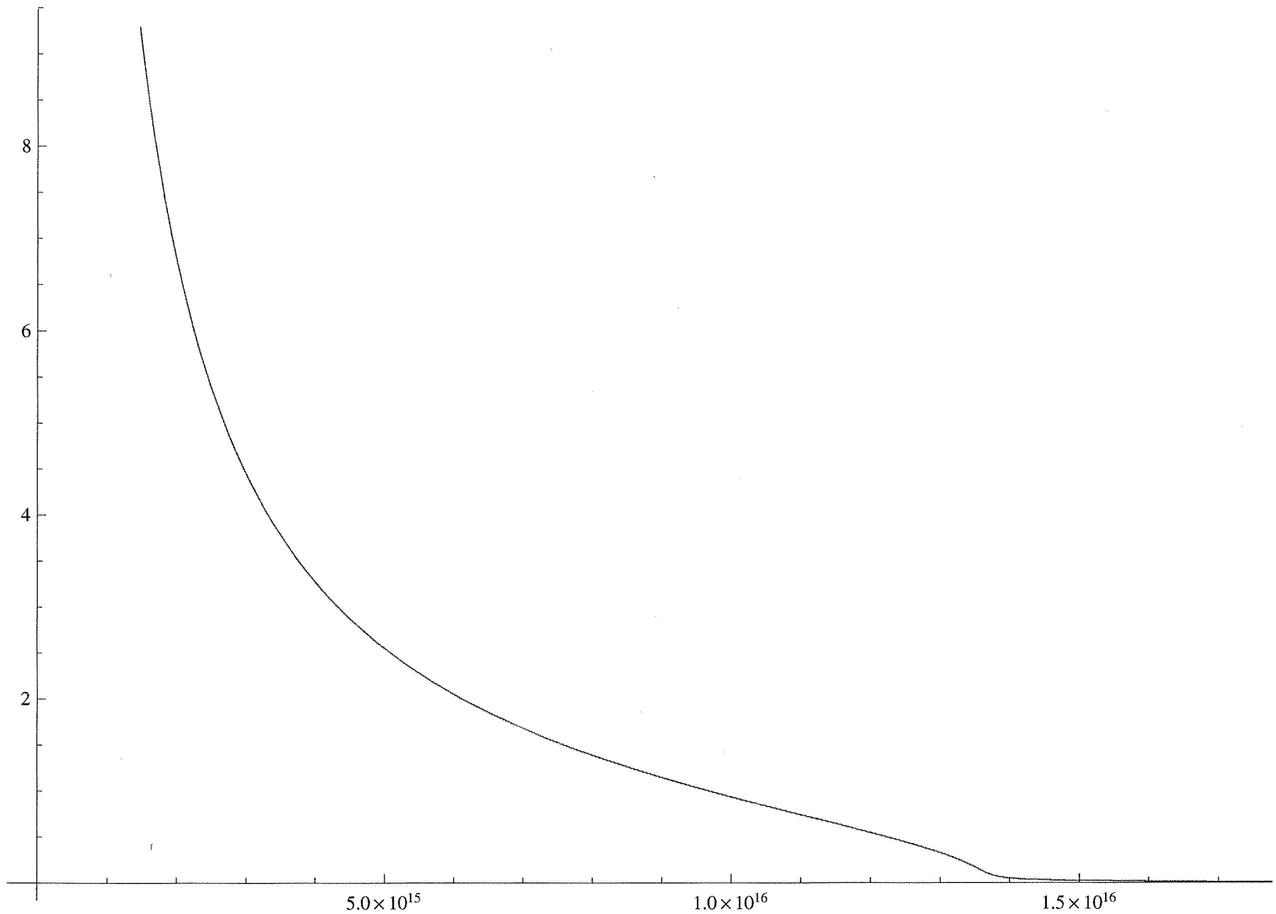
P7

```
(*Refractive index and reflectivity at normal incidence
 as function of frequency for Ag described by the Drude model*)

(*Constants*)
ϵ0 = 8.854187817*^-12;
e = 1.60217646*^-19;
m = 9.10938188*^-31;
σAg = 6.30119722747322 * 10^7;
n = 5.86 * 10^28; (*Reference: Fox*)
(*Drude relaxation time for Ag, γAg= 2.6189*^13 → from σ=
  ne^2/(m(γ-ιω)) → Re[σ=ne^2(γ+ιω)/(γ^2+ω^2)]=ne^2γ/(γ^2+ω^2). Let ω=0 → γ=ne^2/(m*σ)*)
γAg = 2.6189*^13;
(*Plasmon frequency ωP(Ag)=1.3656529121249004`*^16 Hz, Reference: Fox*)
ωp[n_] := Sqrt[n * e^2 / (ϵ0 * m)];
(*electron number density for Ag*)
ndensity[ω_] := (ω^2 * m * ϵ0 / e^2);

(*Drude model permittivity*)
ϵ1[ω_, γ_, N_] := 1 - (N * e^2 / (ϵ0 * m)) * (1 / (ω^2 + γ^2));
ϵ2[ω_, γ_, N_] := (4 π * N * e^2 / (ϵ0 * m)) * γ / (ω * (ω^2 + γ^2));
```

```
(*Index of refraction: N[ω]=
 n+ικ=Sqrt[ϵ]=Sqrt[ϵ1+ιϵ2]→ n=1/Sqrt[2]* Sqrt[Sqrt[ϵ1^2+ϵ2^2]+ϵ1],
κ=Sign[ϵ2]/Sqrt[2]*Sqrt[Sqrt[ϵ1^2+ϵ2^2]-ϵ1]*)
(*Real part*)
nindex[ω_, γ_, N_] := 1 / Sqrt[2] * Sqrt[Sqrt[ϵ1[ω, γ, N]^2 + ϵ2[ω, γ, N]^2] + ϵ1[ω, γ, N]]
Plot[nindex[ω, γAg, n], {ω, 0, 4 * 10^16}]
```

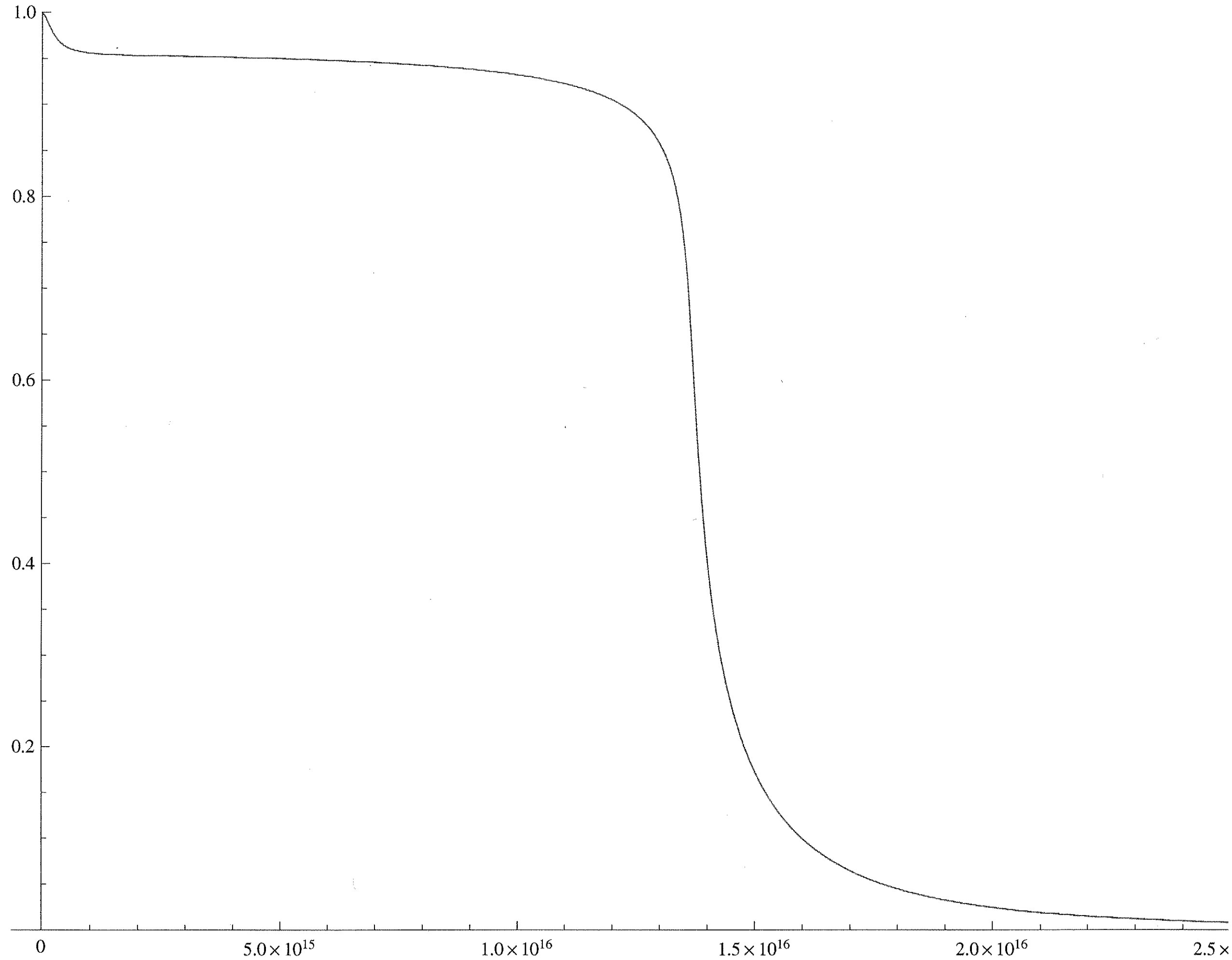


```
(*Imaginary part*)
κ[ω_, γ_, N_] :=
  (Sign[ϵ2[ω, γ, N]] / Sqrt[2]) * Sqrt[Sqrt[ϵ1[ω, γ, N]^2 + ϵ2[ω, γ, N]^2] - ϵ1[ω, γ, N]]
Plot[κ[ω, γAg, n], {ω, 0, 2 * 10^16}]
```




```
(*Reflectivity at normal incidence --> R=|r|^2, r=z-1/z+1, z=(μ/ϵ)^0.5*)
ϵ[ω_, γ_, N_] := ϵ1[ω, γ, N] + i * ϵ2[ω, γ, N];
r[ω_, γ_, N_] := (1 - Sqrt[ϵ[ω, γ, N]]) / (1 + Sqrt[ϵ[ω, γ, N]]);
```

```
Plot[Abs[r[ω, γAg, n]]^2, {ω, 0, 3 * 10^16}, PlotRange → {0, 1}]
```



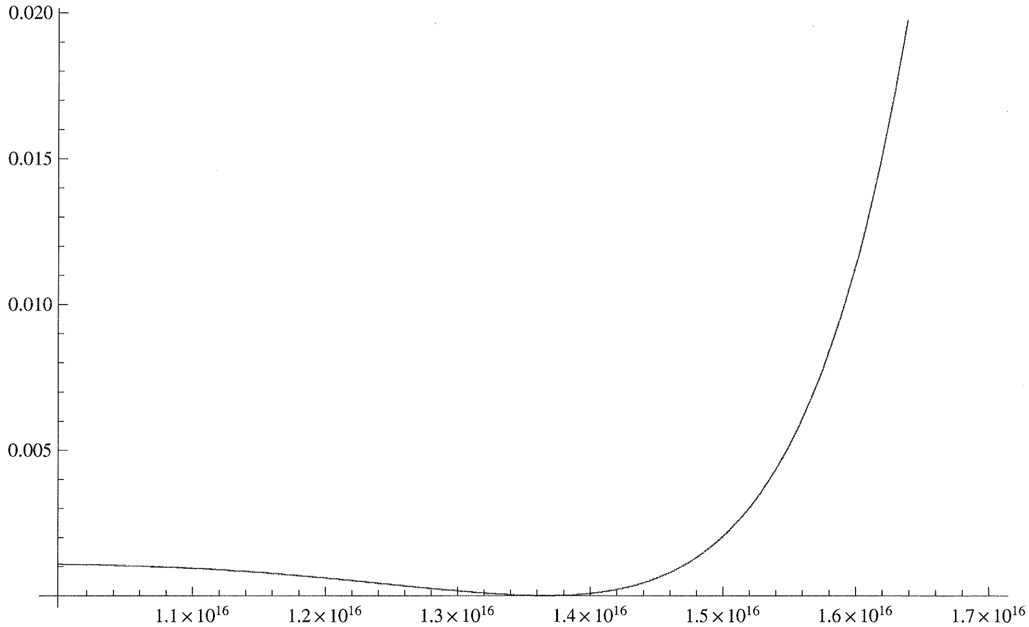

```
(*Compare with experimental values of plasma frequency from Oats & Mucklich,
Nanotechnology 16 (2005).  Note that the plasma frequency obtained here is shifted
  from that experimentally determined.  This is due to the oversimplification
  of the Drude model in which inter-band transitions are not accounted for.*)
```

P8

```
(*Transmission of P-waves through an absorptive medium,
incident at π/4, with permittivity determined by the Drude model*)

(*Transmission Tp(ω) for various material lengths,d, at incident angle of π/4*)

η = d * ω / (2.99*^8);
```

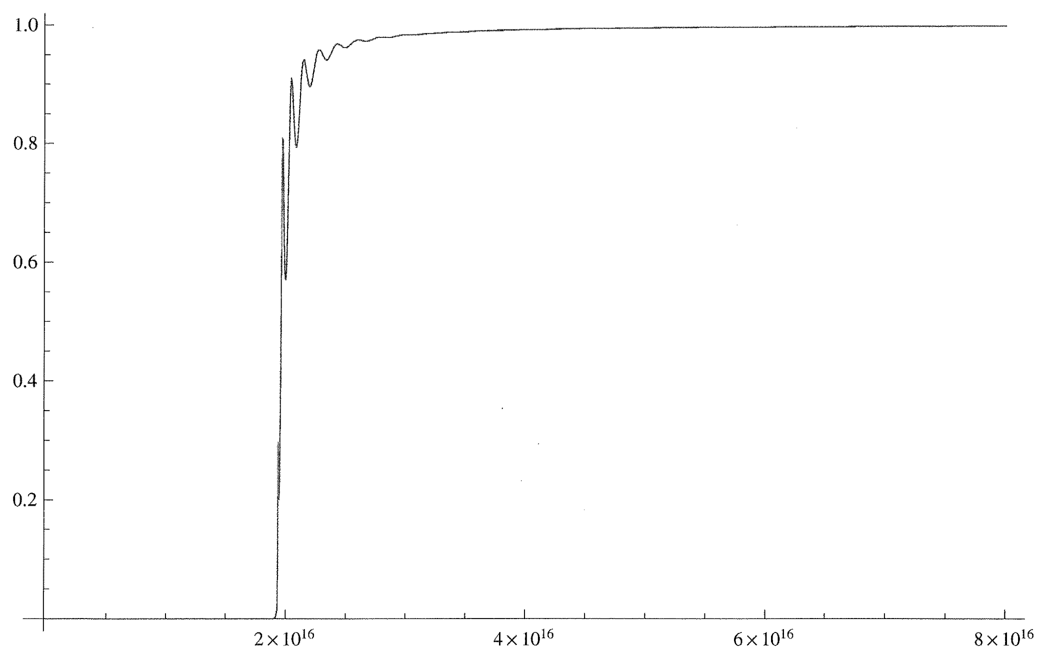
```
Plot[Tp[n, κ, 1, π / 4, π / 4, 1], {ω, 1 * 10^16, 1.7 * 10^16}]
(*d=100nm*)
```

Out[117]=

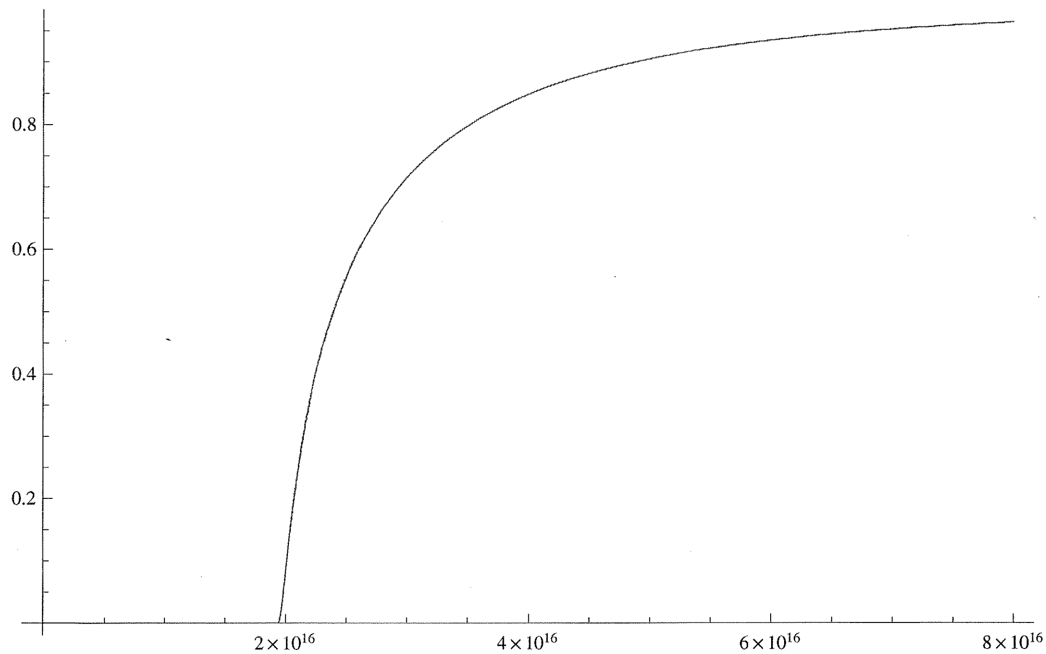



In[113]:=
```
(*d=500nm*)
```

In[114]:=




In[115]:=
```
(*d=10um*)
```




In[116]:=

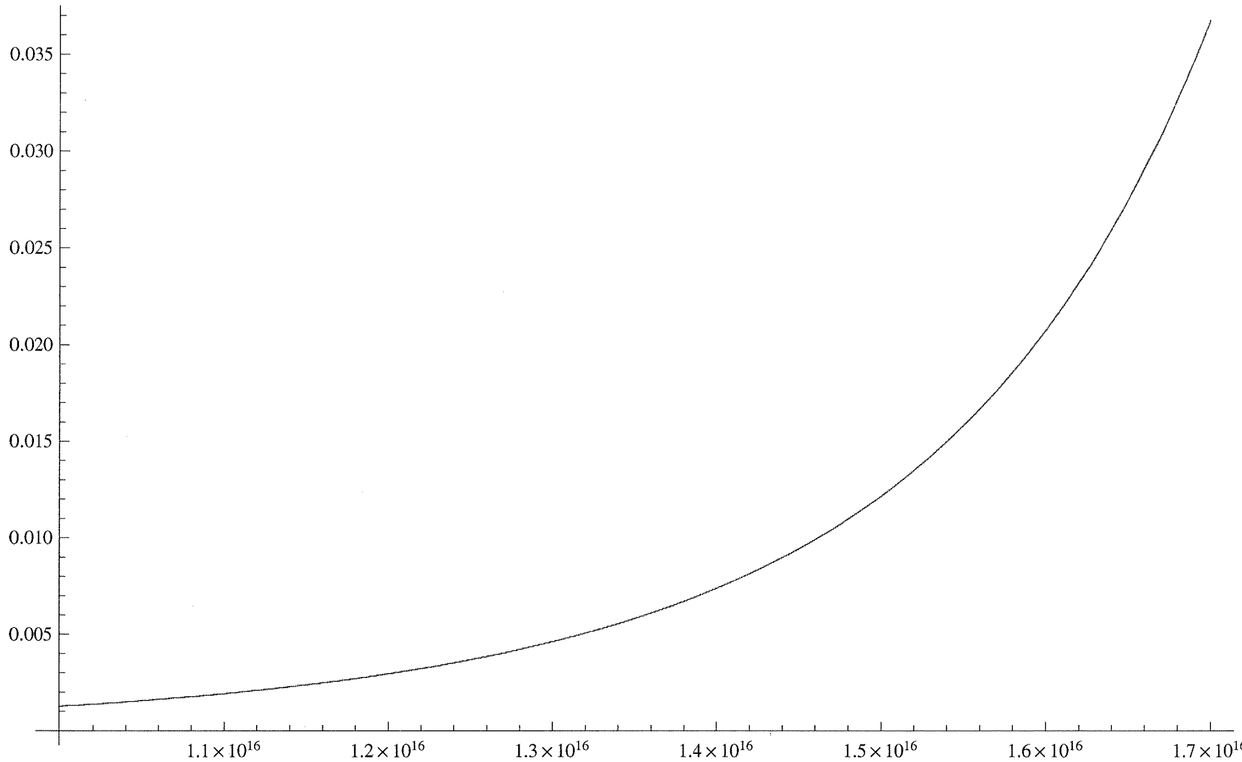
(*Transmission of S-waves through an absorptive medium,
incident at $\pi/4$, with permittivity determined by the Drude model*)

(*Transmission $T_s(\omega)$ for various material lengths,d, at incident angle of $\pi/4$*)

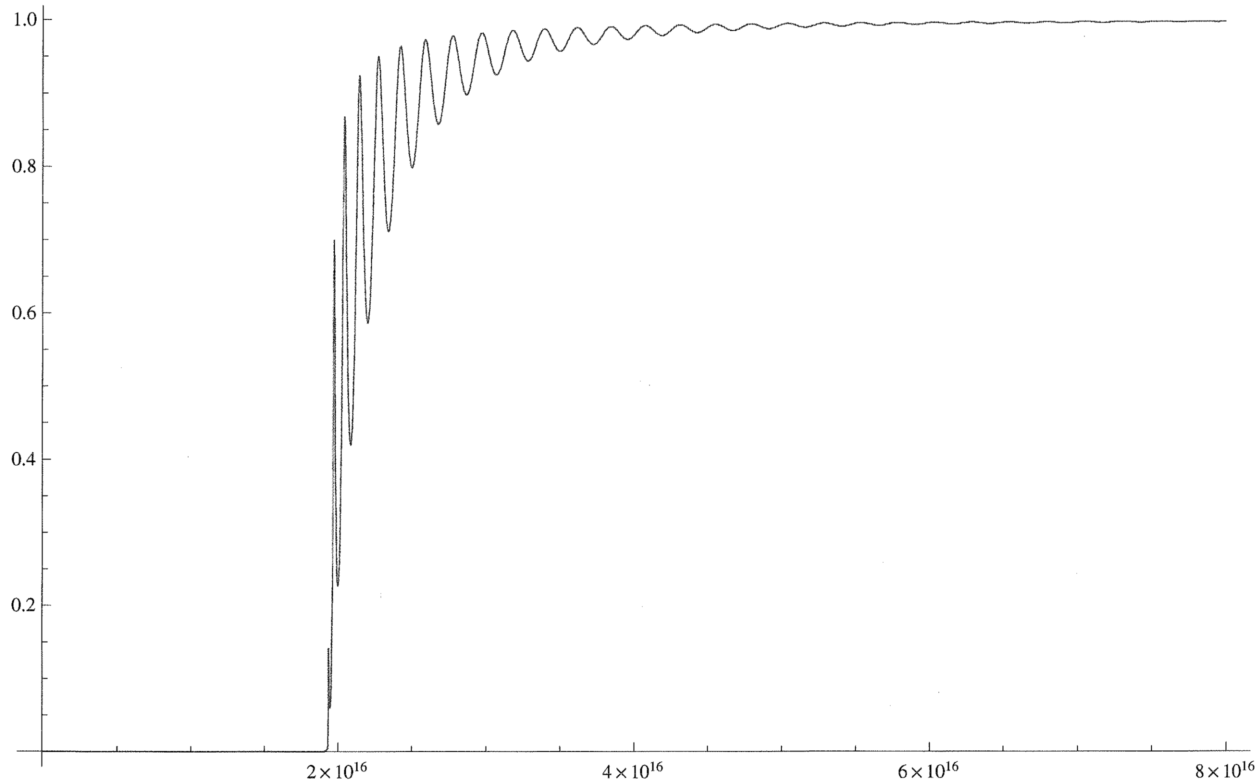
```
η = d * ω / (2.99*^8);
Plot[Ts[n, κ, 1, π / 4, π / 4, 1], {ω, 1 * 10^16, 1.7 * 10^16}]
```

```
(*d=100nm*)
```

Out[85]=



(*d=500nm*)




(*d=10um*)