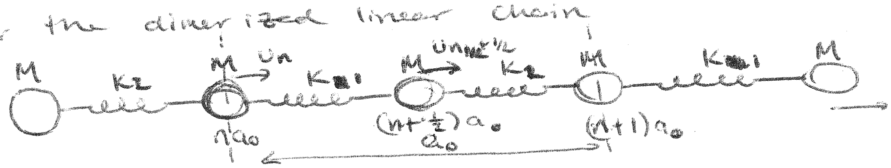
1:

Consider the dimerized linear chain



a) obtain analytical expressions for the phonon dispersion & plot Ω vs. q w/ q the Bloch wavevector.

Let's Define: $F_n = -\sum_{n'} D_{nn'} U_{n'}$

Force on atom at position n

Assuming only nearest neighbor interactions:

$D_{nn} = K_1 + K_2$; $D_{n\,n+1/2} = -K_1$, $D_{n\,n-1/2} = -K_2$ for $R = na_0$

$D_{n\,n+1/2} = -K_2$ $D_{n\,n-1/2} = -K_1$ for $R = (n+\frac{1}{2})a_0$

Then we have the eqns of motion:

(1) $M\ddot{u}_n = -(K_1+K_2)u_n + K_1 U_{n+1/2} + K_2 U_{(n-\frac{1}{2})}$

(2) $M\ddot{U}_{n+1/2} = -(K_1+K_2)u_{n+1/2} + K_2 U_{n+1} + K_1 U_n$

Assume Form of solns:

$U_n(t) = A_1 e^{i(qna_0 - \Omega t)}$; $U_{n+1/2}(t) = A_2 e^{i(qna_0 + qa_0/2 - \Omega t)}$

Plugging in:

(1): $-M\Omega^2 A_1 e^{i(qna_0 - \Omega t)} = -(K_1+K_2)A_1 e^{i(qna_0 - \Omega t)} + K_1 A_2 e^{i(qna_0 + qa_0/2 - \Omega t)} + K_2 A_2 e^{i(qna_0 - qa_0/2 - \Omega t)}$

$\hookrightarrow$ $-M\Omega^2 A_1 = -(K_1+K_2)A_1 + (K_1 e^{iqa_0/2} + K_2 e^{-iqa_0/2})A_2$.

(2): $-M\Omega^2 A_2 e^{i(qna_0 + qa_0/2 - \Omega t)} = -(K_1+K_2)A_2 e^{i(qna_0 + qa_0/2 - \Omega t)} + K_2 A_1 e^{i(qna_0 + qa_0 - \Omega t)} + K_1 A_1 e^{i(qna_0 - \Omega t)}$

$\hookrightarrow$ $-M\Omega^2 A_2 = -(K_1+K_2)A_2 + (K_2 e^{iqa_0/2} + K_1 e^{-iqa_0/2})A_1$

$$\begin{vmatrix} (K_1+K_2) - M\Omega^2 & -\left(K_1 e^{iqa_0/2} + K_2 e^{-iqa_0/2}\right) \\ -\left(K_2 e^{iqa_0/2} + K_1 e^{-iqa_0/2}\right) & (K_1+K_2) - M\Omega^2 \end{vmatrix} = 0$$

$$(K_1+K_2)^2 + M^2\Omega^4 - 2(K_1+K_2)M\Omega^2 - K_1^2 - K_2^2 - K_1K_2\left(e^{iqa_0} + e^{-iqa_0}\right) =$$

$$= M^2\Omega^4 - 2(K_1+K_2)M\Omega^2 + 2K_1K_2 - 2K_1K_2\cos qa_0 = 0$$

Then:

$$\Omega^2 = \frac{1}{M}(K_1+K_2) \pm \left[\frac{1}{M^2}(K_1+K_2)^2 - \frac{2K_1K_2}{M^2}(1-\cos qa_0)\right]^{1/2}.$$

See attached plot Ω vs. q.

b) Calculate numerically the phonon density of states for the acoustic and optical branches.

The Density of States in 1-D is given by:

$$D(w)\,dw = \frac{dN}{dw}\,dw = \frac{dN}{dq}\left|\frac{dq}{dw}\right|dw = \frac{dN}{dq}\frac{1}{|(dw/dq)|}\,dw$$

Using the dispersion relations in (a) we get $\frac{1}{|d\Omega/dq|}$

See attached plot.

c). We assume that the absorption is proportional to the density of states. In the case of 2 phonon absorption we consider the DoS for phonons w/ frequency $\frac{\Omega}{2}$.

See attached plot.

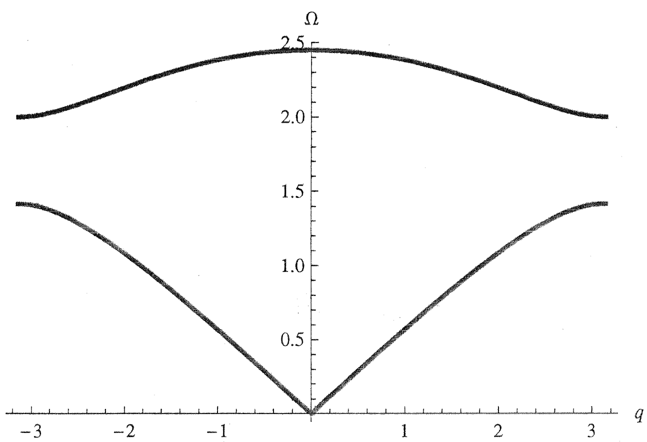
```
(*Problem 3: Part a*)

(*Choosing constants for Plottting purposes*)
k1 = 1;
k2 = 2;
M = 1;

ΩO = Sqrt[(k1 + k2) / M + Sqrt[(k1 + k2)^2 / M^2 - 2 k1 k2 / M^2 (1 - Cos[q])]]
```

$$\sqrt{3+\sqrt{9-4\ (1-\mathrm{Cos}[q])}}$$

```
ΩA = Sqrt[(k1 + k2) / M - Sqrt[(k1 + k2)^2 / M^2 - 2 k1 k2 / M^2 (1 - Cos[q])]]
```

$$\sqrt{3-\sqrt{9-4\ (1-\mathrm{Cos}[q])}}$$

```
Plot[{ΩO, ΩA}, {q, -Pi, Pi}, AxesLabel → {q, Ω}, PlotStyle → Thick]
```



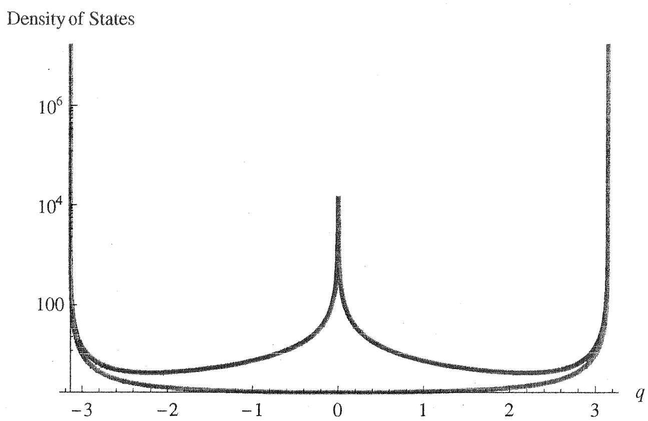

```
(*Part b*)

dΩO = 1 / D[ΩO, q]
```

$$-\sqrt{3+\sqrt{9-4\ (1-\mathrm{Cos}[q])}}\ \sqrt{9-4\ (1-\mathrm{Cos}[q])}\ \mathrm{Csc}[q]$$

```
dΩA = 1 / D[ΩA, q]
```

$$\sqrt{3-\sqrt{9-4\ (1-\mathrm{Cos}[q])}}\ \sqrt{9-4\ (1-\mathrm{Cos}[q])}\ \mathrm{Csc}[q]$$

```
LogPlot[{Abs[Out[7]], Abs[Out[8]]}, {q, -π, π},
 PlotRange → Full, PlotStyle → Thick, AxesLabel → {q, Density of States}]
```




```
DoSO = (1 / (2 * Pi)) * dΩO
```

$$-\frac{\sqrt{3+\sqrt{9-4\,(1-\text{Cos}[q])}}\;\sqrt{9-4\,(1-\text{Cos}[q])}\;\text{Csc}[q]}{2\,\pi}$$

```
DoSA = (1 / (2 * Pi)) * dΩA
```

$$\frac{\sqrt{3-\sqrt{9-4\,(1-\text{Cos}[q])}}\;\sqrt{9-4\,(1-\text{Cos}[q])}\;\text{Csc}[q]}{2\,\pi}$$

```
(*Writing DoS as a function of Ω's to plot and taking absolute values:*)
```

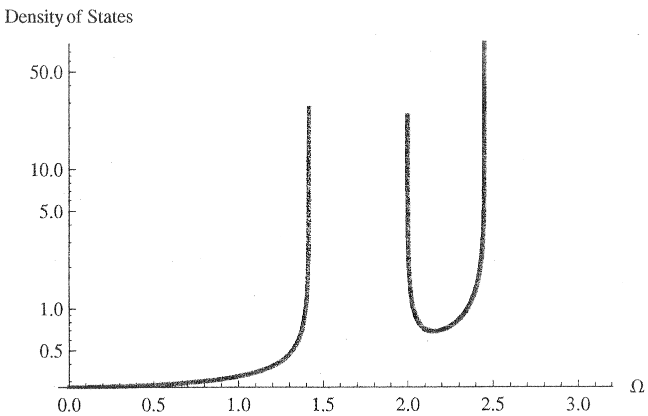
$$\text{DoSO} = \frac{1}{2\,\pi}\,\Omega\,(\Omega\hat{}2-3)\;\text{Csc}[\text{ArcCos}[(5-(\Omega\hat{}2-3)\hat{}2)/4]]$$

$$\frac{\Omega\left(-3+\Omega^2\right)}{2\,\pi\sqrt{1-\frac{1}{16}\left(5-\left(-3+\Omega^2\right)^2\right)^2}}$$

$$\text{DoSA} = \frac{1}{2\,\pi}\,\Omega\,(3-\Omega\hat{}2)\;\text{Csc}[\text{ArcCos}[(5-(3-\Omega\hat{}2)\hat{}2)/4]]$$

$$\frac{\Omega\left(3-\Omega^2\right)}{2\,\pi\sqrt{1-\frac{1}{16}\left(5-\left(3-\Omega^2\right)^2\right)^2}}$$

```
LogPlot[{DoSO, DoSA}, {Ω, 0, π}, PlotRange → Full,
 PlotStyle → Thick, AxesLabel → {Ω, Density of States}]
```



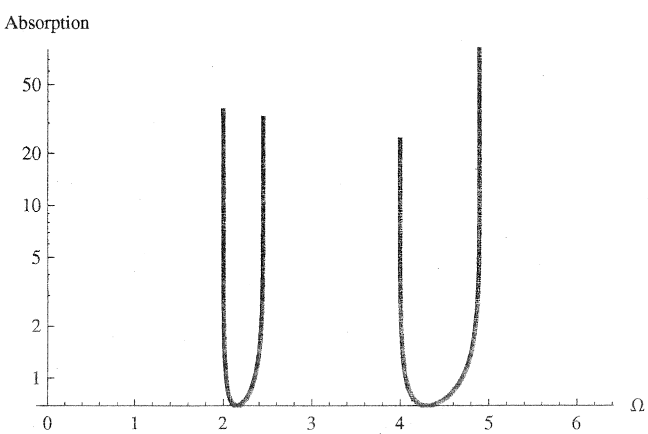

```
(*Part c*)

α1 = DoSO
```

$$\frac{\Omega\left(-3+\Omega^2\right)}{2\pi\sqrt{1-\frac{1}{16}\left(5-\left(-3+\Omega^2\right)^2\right)^2}}$$

```
α2 = DoSO /. Ω → Ω / 2
```

$$\frac{\Omega\left(-3+\frac{\Omega^2}{4}\right)}{4\pi\sqrt{1-\frac{1}{16}\left(5-\left(-3+\frac{\Omega^2}{4}\right)^2\right)^2}}$$

```
LogPlot[{α1, α2}, {Ω, 0, 2 π}, PlotRange → Full, PlotStyle → Thick, AxesLabel → {Ω, Absorption}]
```



2) a) The general equations of motion that relate $\bar{E}, \bar{P}$ and ion displacement $\bar{W}$ are:

$$\frac{d^2\bar{W}}{dt^2} = b_{11}\bar{W} + b_{12}\bar{W}$$

(For diatomic crystals. $b_{12} = b_{21}$)

$$\bar{P} = b_{21}\bar{W} + b_{22}\bar{E}$$

where the coefficients $b_{11}, b_{12}, b_{21}, b_{22}$ are in general tensors (depend on direction of $\bar{E}, \bar{P}, \bar{W}$).

Along principal axes,

$$\bar{P} = \begin{pmatrix} \chi_{11} & & 0 \\ & \chi_{22} & \\ 0 & & \chi_{33} \end{pmatrix} \bar{E}$$

and therefore equations of lattice motion can be decoupled into three equations along principal axes

$$\frac{d^2 W_i}{dt^2} = b_{11_i} W_i + b_{12_i} E_i$$

(where $i = x, y, z$ principal axes)

$$P_i = b_{21_i} W_i + b_{22_i} E_i$$

Now, we know that E-mode corresponds to dipole moments along x, y principal axes. The equations become,

$$\frac{d^2 W_{(x,y)}}{dt^2} = b_{11(x,y)} W_{x,y} + b_{12(x,y)} E_{x,y}$$

$$P_{x,y} = b_{21(x,y)} W_{x,y} + b_{22(x,y)} E_{x,y}$$

Assuming time dependence $e^{i\omega t}$ these can be solved to relate

$$P_{x,y} = \chi_{x,y} E_{x,y}, \quad \epsilon_{x,y} = 1 + 4\pi \chi_{x,y}$$

As done in class, the form of dielectric constant is

$$\boxed{\epsilon_{x,y} = \epsilon_\infty \left(\frac{\Omega_{ELO}^2 - \Omega^2}{\Omega_{ETO}^2 - \Omega^2} \right)}$$

where $\Omega_{ETO}^2 = -b_{11}(x,y)$

$$\epsilon_\infty = 1 + 4\pi b_{22}(x,y)$$

$$\Omega_{ELO}^2 = \Omega_{ETO}^2 + \frac{4\pi b_{12}(x,y)}{\epsilon_\infty}$$

Similarly, A-mode is responsible for displacements along z-axis. Thus, the equations of motions are

$$\frac{d^2 W_z}{dt^2} = b_{11z} W_z + b_{12z} E_z$$

$$P_z = b_{21z} W_z + b_{22z} E_z$$

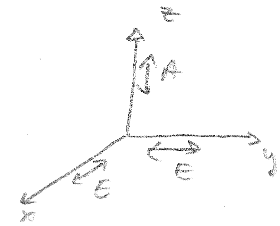
which yield,

$$\boxed{\epsilon_z = \epsilon_\infty \left(\frac{\Omega_{ALO}^2 - \Omega^2}{\Omega_{ATO}^2 - \Omega^2} \right)}$$

where $\Omega_{ATO}^2 = -b_{11z}$

$$\epsilon_\infty = 1 + 4\pi b_{22z}$$

$$\Omega_{ALO}^2 = \Omega_{ATO}^2 + \frac{4\pi b_{12z}}{\epsilon_\infty}$$



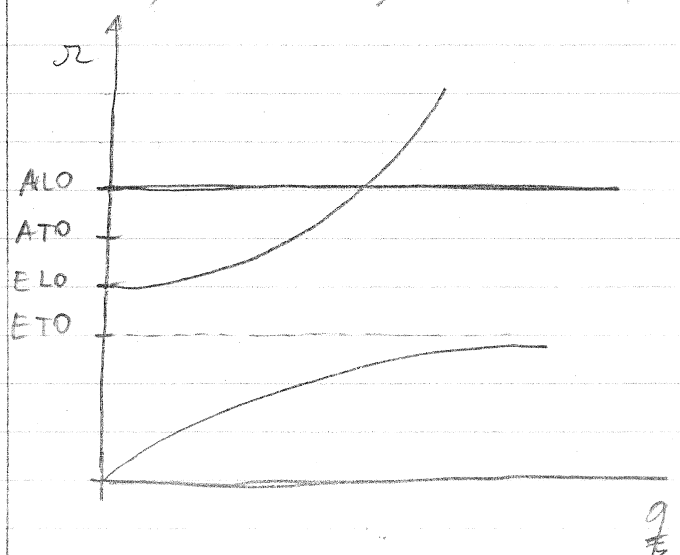
b) Plot dispersion relations for $\underline{q} \parallel x$ and $\underline{q} \parallel z$ in a uniaxial crystal.

$\underline{\bar{q}} \parallel \bar{z}$:

Since $\bar{q} \parallel z$, it is possible to excite the following displacements:

- Displacements parallel to $\bar{z}$ (A mode), which would have to be longitudinal since $\underline{\bar{q}} \parallel \bar{z}$.
- Displacements along $\bar{x}$ or $\bar{y}$ (both $\bar{E}$ mode), which would have to be transverse.

Assuming A mode (both LO and TO) lies above E mode, we get the following.



$\underline{\bar{q}} \parallel \bar{x}$

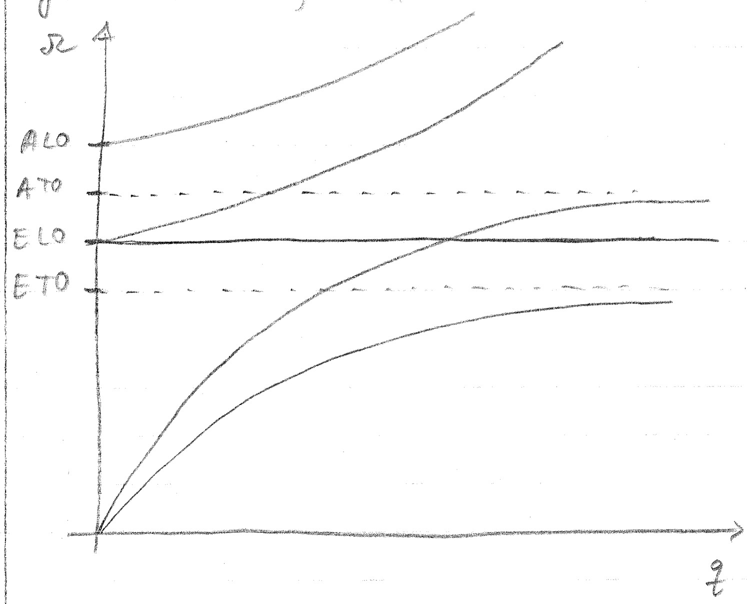
For this case there are two eigenmodes allowed in a uniaxial crystal — $\bar{D}_o$ and $\bar{D}_e$ (ordinary and extraordinary ray).

- Ordinary ray can excite transverse displacements of E-mode along y-axis.

- Extraordinary ray can excite lattice displacements along z-axis, which would be transverse and belong to A.

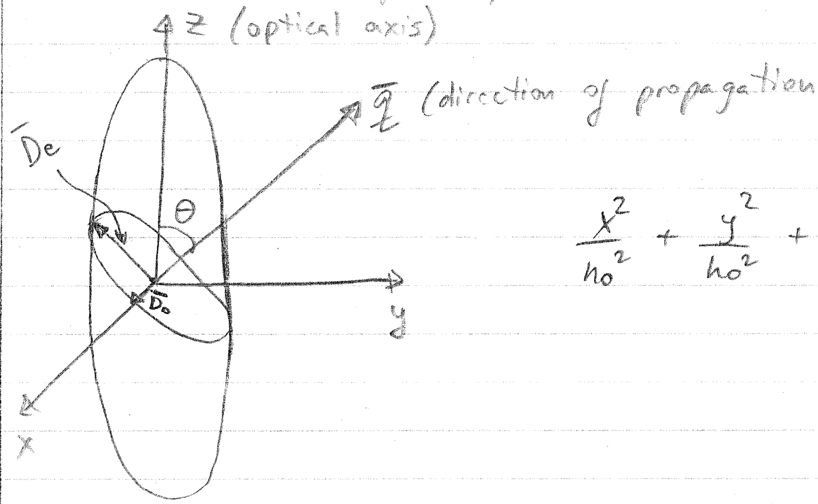
- In addition $\bar{q} \parallel \bar{x}$ can excite longitudinal displacements along $\bar{x}$ (ELO).

Again assuming all A modes are above E modes we get,



e) Refractive indices for the two orthogonal polarization modes in a uniaxial crystal may be found using method of the Index Ellipsoid (Reference: Optical Waves in Crystals by Yariv, Yeh).

It works as follows,



$$\frac{x^2}{n_o^2} + \frac{y^2}{n_o^2} + \frac{z^2}{n_e^2} = 1$$

Any direction of propagation in a uniaxial crystal allows two orthogonal polarizations – $\bar{D}_o, \bar{D}_e$ – that experience different indices of refraction. The indices of refraction are found from the principal axes of an ellipse formed by intersection of Index Ellipsoid with the plane perpendicular to $\bar{q}$ (propagation direction) going through the center of Index Ellipsoid.

As seen from figure above, the principal axis corresponding to $\bar{D}o$ is n_o. The principal axis for $\bar{D}e$ is found by trigonometry and using equation for the ellipsoid (in the z-y plane for simplicity),

$$\frac{x^2}{n_o^2} + \frac{y^2}{n_o^2} + \frac{z^2}{n_e^2} = 1$$

$$n_e(\theta)\cos\theta = x$$
$$n_e(\theta)\sin\theta = z$$
$$0 = y$$

$$\frac{[n_e(\theta)\cos\theta]^2}{n_o^2} + \frac{[n_e(\theta)\sin\theta]^2}{n_e^2} = 1$$

$$\frac{1}{n_e^2(\theta)} = \frac{\cos^2\theta}{n_o^2} + \frac{\sin^2\theta}{n_e^2}$$

In our case $n_o^2 = \epsilon_{x,y}$, $n_e^2 = \epsilon_z$, $n_e(\theta) = \frac{q c}{\Omega}$ (From dispersion relation)

$$\Rightarrow \boxed{\frac{\Omega^2}{q^2 c^2} = \frac{\sin^2\theta}{\epsilon_z(\Omega)} + \frac{\cos^2\theta}{\epsilon_{x,y}(\Omega)}}$$

In order to plot Ω vs θ for extraordinary ray, must solve for zeros of the function

$$\frac{\Omega^2}{q^2c^2} = \frac{\sin^2\theta}{\epsilon_\infty}\left(\frac{\Omega_{ATO}^2 - \Omega^2}{\Omega_{ALO}^2 - \Omega^2}\right) + \frac{\cos^2\theta}{\epsilon_\infty}\left(\frac{\Omega_{ETO}^2 - \Omega^2}{\Omega_{ELO}^2 - \Omega^2}\right)$$

$$0 = \frac{\epsilon_\infty \Omega^2 (\Omega_{ALO}^2 - \Omega^2)(\Omega_{ELO}^2 - \Omega^2)}{q^2c^2} + \sin^2\theta(\Omega_{ATO}^2 - \Omega^2)(\Omega_{ELO}^2 - \Omega^2)$$
$$+ \cos^2\theta(\Omega_{ETO}^2 - \Omega^2)(\Omega_{ALO}^2 - \Omega^2)$$

As $q \to \infty$,

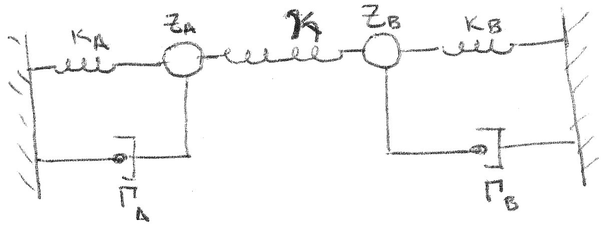
$$0 = \sin^2\theta(\Omega_{ATO}^2 - \Omega^2)(\Omega_{ELO}^2 - \Omega^2) + \cos^2\theta(\Omega_{ETO}^2 - \Omega^2)(\Omega_{ALO}^2 - \Omega^2)$$

(excluding points $\Omega = \Omega_{ALO}$, $\Omega = \Omega_{ELO}$).

$$\Omega^4(\sin^2\theta + \cos^2\theta) - \left(\sin^2\theta(\Omega_{ATO}^2 + \Omega_{ELO}^2) + \cos^2\theta(\Omega_{ETO}^2 + \Omega_{ALO}^2)\right)\Omega^2$$
$$+ \sin^2\theta\, \Omega_{ATO}^2 \Omega_{ELO}^2 + \cos^2\theta\, \Omega_{ETO}^2 \Omega_{ALO}^2 = 0$$

In addition, for $q \to 0$ there is $\Omega = E_{TO}$ solution for ordinary ray.

5 :



$$U_c = \frac{\mathcal{K}(u_A - u_B)^2}{2}$$

a) Calculate the polarization $P = eZ_A u_A + eZ_B u_B$ induced by an electric field of freq. ω and magnitude E. Find the real and imaginary part of the permittivity.

Eqns of Motion:

$$\ddot{u}_1 + (K_1 + \mathcal{K})u_1 + \Gamma_1 \dot{u}_1 - \mathcal{K} u_2 = Z_1 e E$$

$$\ddot{u}_2 + (K_2 + \mathcal{K})u_2 + \Gamma_2 \dot{u}_2 - \mathcal{K} u_1 = Z_2 e E$$

Taking $E = E_0 e^{-i\omega t}$ & assuming $u_1 \propto U_2 \propto e^{-i\omega t}$

$$-\omega^2 U_1 + (K_1 + \mathcal{K})U_1 - i\omega \Gamma_1 U_1 - \mathcal{K} U_2 = Z_1 e E_0$$

$$-\omega^2 U_2 + (K_2 + \mathcal{K})U_2 - i\omega \Gamma_2 U_2 - \mathcal{K} U_1 = Z_2 e E_0$$

After eliminating time dependence.

Solving: Set: $-\omega^2 + (K_i + \mathcal{K}) - i\omega\Gamma_i = \Sigma_i$

$$U_1 = \frac{\mathcal{K} U_2 + Z_1 e E_0}{\Sigma_1} \longrightarrow \Sigma_2 U_2 - \mathcal{K}\left(\frac{\mathcal{K} U_2 + Z_1 e E_0}{\Sigma_1}\right) = Z_2 e E_0$$

$$\hookrightarrow U_2 (\Sigma_1 \Sigma_2 - \mathcal{K}^2) = (\Sigma_1 Z_2 + \mathcal{K} Z_1) e E_0$$

$$\therefore U_2 = \frac{\mathcal{K} Z_1 + \Sigma_1 Z_2}{\Sigma_1 \Sigma_2 - \mathcal{K}^2} e E_0$$

Similarly:

$$U_1 = \frac{\mathcal{K} Z_2 + \Sigma_2 Z_1}{\Sigma_1 \Sigma_2 - \mathcal{K}^2} e E_0 .$$

Then:

$$P = \frac{\Sigma_1 Z_2^2 + \Sigma_2 Z_1^2 + 2\mathcal{K} Z_1 Z_2}{\Sigma_1 \Sigma_2 - \mathcal{K}^2} e^2 E_0 .$$

Note:

$$\epsilon \vec{E} - \vec{E} = 4\pi \vec{P}$$

$$\hookrightarrow \epsilon = 1 + \frac{4\pi \vec{P}}{\vec{E}}$$

b) Performing a linear unitary transformation:

$$\begin{pmatrix} w_A \\ w_B \end{pmatrix} = \underbrace{\begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}}_{R} \begin{pmatrix} u_A \\ u_B \end{pmatrix} \quad \text{w/ } \cot^2\theta + (k_A - k_B)\frac{\cot\theta}{\kappa} = 1$$

Transforming force eqns:

$$\frac{d^2}{dt^2} R \begin{pmatrix} u_A \\ u_B \end{pmatrix} + \frac{d}{dt} R \begin{pmatrix} \Gamma_A & 0 \\ 0 & \Gamma_B \end{pmatrix} R^{-1} R \begin{pmatrix} u_A \\ u_B \end{pmatrix} + R \begin{pmatrix} k_A + \kappa & -\kappa \\ -\kappa & k_B + \kappa \end{pmatrix} R^{-1} R \begin{pmatrix} u_A \\ u_B \end{pmatrix} =$$

$$= R \begin{pmatrix} z_A \\ z_B \end{pmatrix} e E_0$$

Then we see:

$$R \begin{pmatrix} k_A + \kappa & -\kappa \\ -\kappa & k_B + \kappa \end{pmatrix} R^{-1} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} k_A + \kappa & -\kappa \\ -\kappa & k_B + \kappa \end{pmatrix} \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}.$$

This will be diagonalized for 0 off diagonal terms:

$$-(k_A + \kappa)\cos\theta\sin\theta + \kappa\sin^2\theta - \kappa\cos^2\theta + (k_B + \kappa)\sin\theta\cos\theta = 0.$$

$$\hookrightarrow (k_A - k_B)\cos\theta\sin\theta + \kappa(\sin^2\theta - \cos^2\theta) = 0$$

which holds for $\cot^2\theta + (k_A - k_B)\dfrac{\cot\theta}{\kappa} = 1.$

$$\Rightarrow R \begin{pmatrix} k_A + \kappa & -\kappa \\ -\kappa & k_B + \kappa \end{pmatrix} R^{-1} = \begin{pmatrix} k_A & 0 \\ 0 & k_B \end{pmatrix}$$

Then considering:

$$R \begin{pmatrix} \Gamma_A & 0 \\ 0 & \Gamma_B \end{pmatrix} R^{-1} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} \Gamma_A & 0 \\ 0 & \Gamma_B \end{pmatrix} \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} =$$

$$= \begin{pmatrix} \Gamma_A\cos^2\theta + \Gamma_B\sin^2\theta & \sin\theta\cos\theta(\Gamma_B - \Gamma_A) \\ \sin\theta\cos\theta(\Gamma_B - \Gamma_A) & \Gamma_A\sin^2\theta + \Gamma_B\cos^2\theta \end{pmatrix} = \begin{pmatrix} \gamma_{11} & \gamma_{12} \\ \gamma_{21} & \gamma_{22} \end{pmatrix}$$

Then the force equations become

$$\frac{d^2}{dt^2} \begin{pmatrix} w_A \\ w_B \end{pmatrix} + \frac{d}{dt} \begin{pmatrix} \gamma_{11} & \gamma_{12} \\ \gamma_{21} & \gamma_{22} \end{pmatrix} \begin{pmatrix} w_A \\ w_B \end{pmatrix} + \begin{pmatrix} k_A & 0 \\ 0 & k_B \end{pmatrix} \begin{pmatrix} w_A \\ w_B \end{pmatrix} = R \begin{pmatrix} z_A \\ z_B \end{pmatrix} e E_0$$

which is the form for 2 coupled oscillators w/ interaction damping.

```
(*Problem 5: Part a*)

Σ1 = -ω^2 + (k1 + κ) - ⅈ ω Γ1
```

$k1+\kappa-i\,\Gamma 1\,\omega-\omega^2$

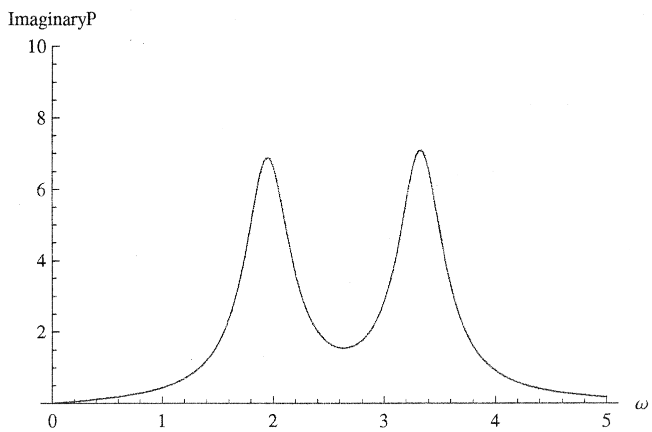
```
Σ2 = -ω^2 + (k2 + κ) - ⅈ ω Γ2
```

$k2+\kappa-i\,\Gamma 2\,\omega-\omega^2$

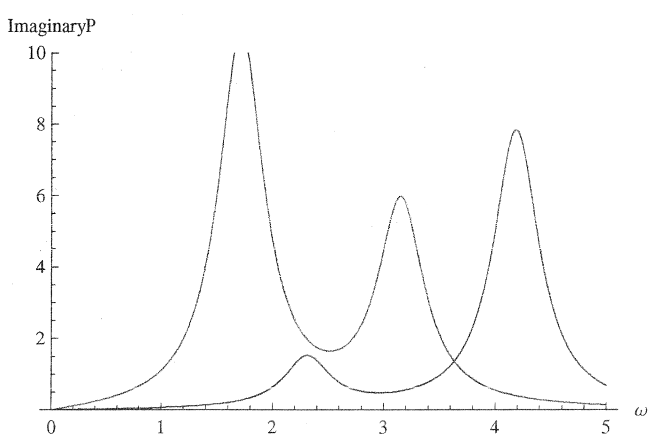
```
P = (Σ1 Z2^2 + Σ2 Z1^2 + 2 κ Z1 Z2) e E0 / (Σ1 Σ2 - κ^2)
```

$$\frac{e\,E0\left(2\,Z1\,Z2\,\kappa+Z2^2\left(k1+\kappa-i\,\Gamma 1\,\omega-\omega^2\right)+Z1^2\left(k2+\kappa-i\,\Gamma 2\,\omega-\omega^2\right)\right)}{-\kappa^2+\left(k1+\kappa-i\,\Gamma 1\,\omega-\omega^2\right)\left(k2+\kappa-i\,\Gamma 2\,\omega-\omega^2\right)}$$

```
ImP = Im[P];
ReP = Re[P];

(*Part c*)
Plot[ImP /. {k1 → 10, k2 → 3, m → 1, κ → 1, Γ1 → .5, Γ2 → .5, e → 1, E0 → 1, Z1 → 3, Z2 → -3},
 {ω, 0, 5}, PlotRange → {0, 10}, AxesLabel → {ω, ImaginaryP}]
```

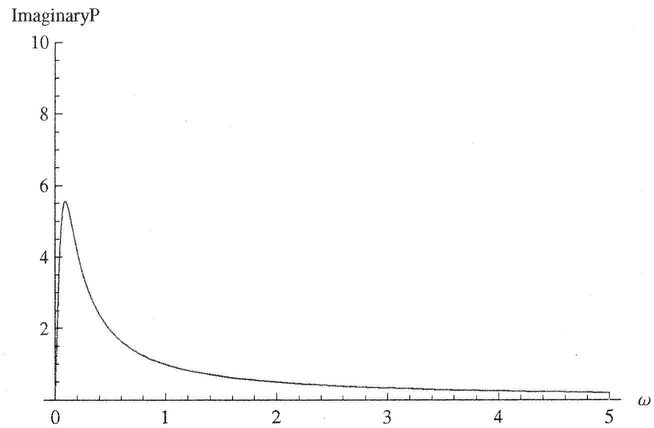



```
(*Note the minimum between the two resonant frequencies*)

Plot[{ImP /. {k1 → 10, k2 → 3, m → 1, κ → 5, Γ1 → .5, Γ2 → .5, e → 1, E0 → 1, Z1 → 3, Z2 → -3},
  ImP /. {k1 → 10, k2 → 3, m → 1, κ → 0, Γ1 → .5, Γ2 → .5, e → 1, E0 → 1, Z1 → 3, Z2 → -3}},
 {ω, 0, 5}, PlotRange → {0, 10}, AxesLabel → {ω, ImaginaryP}]
```

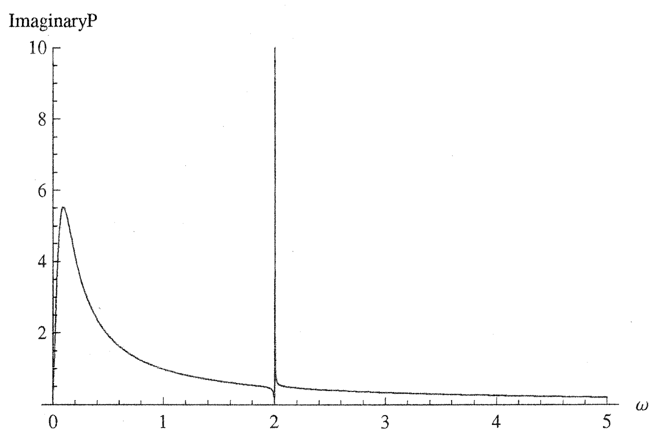



```
(*Note with coupling the minimum is lower than w/o coupling*)
```

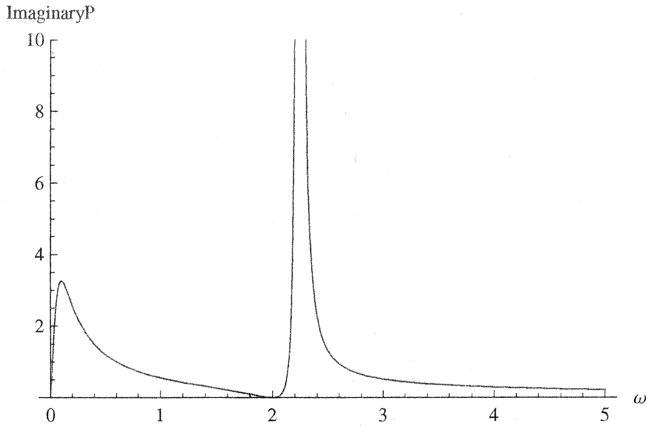
```
(*Part d: Case for 1 overdamped and 1 w/ no damping*)
(*No Coupling*)
Plot[ImP /. {k1 → 9, k2 → 4, m → 1, κ → 0, Γ1 → 100, Γ2 → 0, e → 1, E0 → 1, Z1 → 10, Z2 → -10},
 {ω, 0, 5}, PlotRange → {0, 10}, AxesLabel → {ω, ImaginaryP}]
```




```
(*Adding Coupling*)
Plot[
 ImP /. {k1 → 9, k2 → 4, m → 1, κ → 0.01, Γ1 → 100, Γ2 → 0, e → 1, E0 → 1, Z1 → 10, Z2 → -10},
 {ω, 0, 5}, PlotRange → {0, 10}, AxesLabel → {ω, ImaginaryP}]
```



```
(*Increasing Coupling*)
Plot[ImP /. {k1 → 9, k2 → 4, m → 1, κ → 1, Γ1 → 100, Γ2 → 0, e → 1, E0 → 1, Z1 → 10, Z2 → -10},
 {ω, 0, 5}, PlotRange → {0, 10}, AxesLabel → {ω, ImaginaryP}]
```




```
(*Note the imaginary part drops to zero exhibiting Fano Interference*)
```