Solution

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(1) Thomas - Fermi screening is an approximation to RPA (Lindhard Theory of Screening) when k is small under static limit ($\omega = 0$),

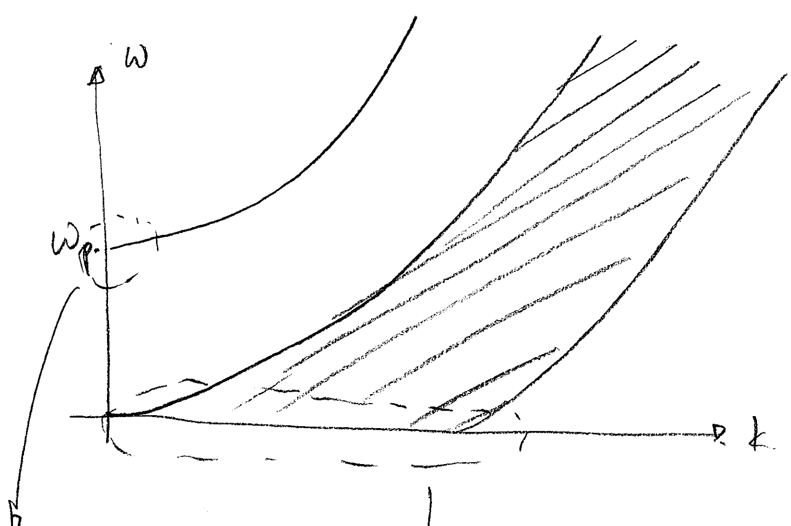
$$\epsilon_{RPA}(k) = 1 + \frac{k_0^2}{k^2}\left\{ \frac{1}{2} + \frac{k_F}{2k}\left(1 - \left(\frac{k}{2k_F}\right)^2\right) \ln\left|\frac{2k_F + k}{2k_F - k}\right| \right\}.$$

where

$$\lim_{k \to 0} \left\{ \frac{1}{2} + \frac{k_F}{2k}\left(1 - \left(\frac{k}{2k_F}\right)^2\right) \ln\left|\frac{2k_F + k}{2k_F - k}\right| \right\} = 1.$$

$$= \epsilon_{\text{Thomas - Fermi}}$$

This approximation is valid only when $\omega = 0$, it is not valid in the regime when plasmons occur ($\omega > 0$)



$\omega = 0 \Rightarrow \epsilon = 1 + \frac{k_F^2}{k^2}$

$k = 0$, $\omega \neq 0$

$\epsilon = 1 - \frac{\omega_p^2}{\omega^2}$

PROBLEM 3

①

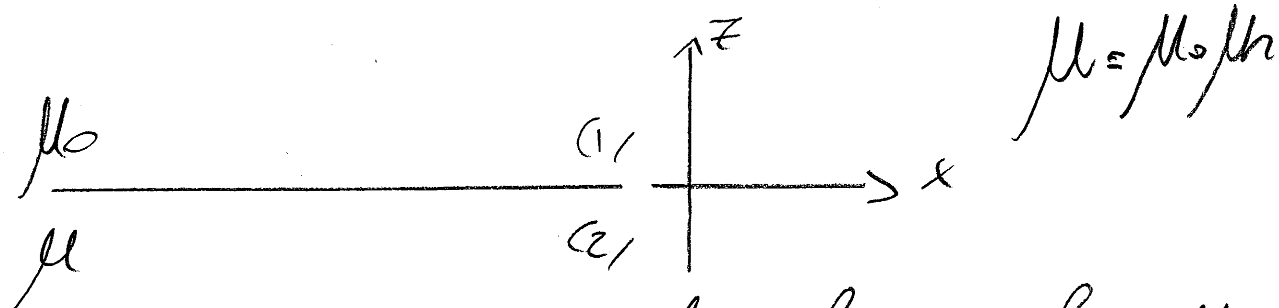
For magnetostatic we have 2 equations:

$$\vec{\nabla}\cdot\vec{B}=0 \qquad \vec{\nabla}\times\vec{B}=\mu\vec{J}$$

Since we don't have current densities in our case ($\vec{J}=0$) the second equation becomes $\vec{\nabla}\times\vec{B}=0$. This allows us to write $\vec{B}=-\vec{\nabla}\phi$ and to recover Laplace eqn:

$$\nabla^2\phi=0$$

We will try to solve this equation for the prototype case of an interface between vacuum and a medium with permeability μ:



We use the same kind of solution form as for our electric plasmon: exponentially decaying function on both sides

$$z>0 \qquad \phi=B^{+}e^{-\lambda z}\cdot e^{ikx}$$

$$z<0 \qquad \phi=B^{-}e^{\lambda z}\cdot e^{ikx}$$

The boundary conditions are different now: using $\vec{\nabla}\times\vec{H}=0$ and $\vec{\nabla}\cdot\vec{B}=0$ we get

$$\begin{cases} \vec{B_1} \cdot \hat{m} = \vec{B_2} \cdot \hat{m} \\ \vec{H_1} \times \hat{m} = \vec{H_2} \times \hat{m} \end{cases}$$

$\hat{m}$ being the normal to the interface. This gives:

$$+B^+ \cdot 2 = -B^- \cdot 2 \quad \Rightarrow \quad B^+ = -B^-$$

$$B^+ \cdot \frac{ik}{\mu_0} = B^- \cdot \frac{ik}{\mu} \quad \Rightarrow \quad -1 = \frac{1}{\mu_r}$$

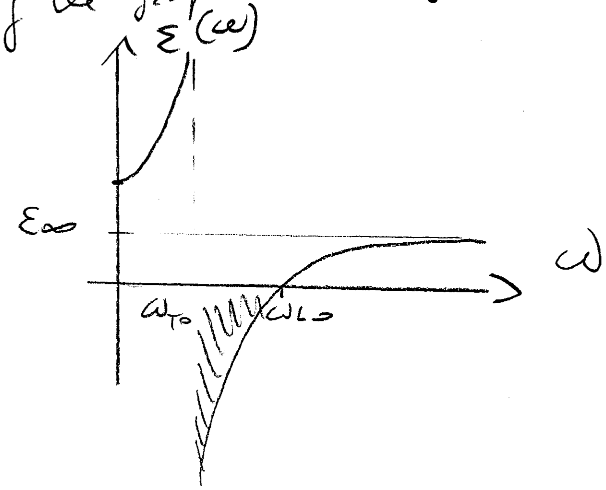
So to satisfy the B.C. we have to require $\mu_r = -1$.
It's very difficult, to have $\mu = -1$ in natural materials and this is the reason why there aren't magnetic plasmon.

PROBLEM 4

The permittivity associated with infrared active modes is:

$$\varepsilon = \varepsilon_\infty \frac{\omega_{LO}^2 - \omega^2}{\omega_{TO}^2 - \omega^2}$$

If we graph it we get:



So there is an area $\omega_{TO} < \omega < \omega_{LO}$ where the permittivity is negative; this means that surface plasmon modes are supported.

We use the expression derived in class:

$$\sqrt{\frac{K^2 + |\varepsilon|\omega^2/c^2}{K^2 - \omega^2/c^2}} = |\varepsilon|$$

We now try to calculate the dispersion relation: ω vs K

$$K^2|\varepsilon|^2 - \frac{\omega^2}{c^2}|\varepsilon|^2 = K^2 + |\varepsilon|\frac{\omega^2}{c^2}$$

$$K^2\varepsilon_\infty^2 \frac{\omega_{LO}^4 + \omega^4 - 2\omega^2\omega_{LO}^2}{\omega_{TO}^4 + \omega^4 - 2\omega^2\omega_{TO}^2} - \frac{\omega^2}{c^2}\varepsilon_\infty^2 \frac{\omega_{LO}^4 + \omega^4 - 2\omega^2\omega_{LO}^2}{\omega_{TO}^4 + \omega^4 - 2\omega^2\omega_{TO}^2} =$$

$$K^2 + \frac{\omega^2}{c^2}\frac{\omega^2 - \omega_{LO}^2}{\omega_{TO}^2 - \omega^2}$$

$$K^2\varepsilon_\infty^2\left(\omega_{LO}^4+\omega^4-2\omega^2\omega_{LO}^2\right)-\frac{\omega^2}{c^2}\varepsilon_\infty^2\left(\omega_{LO}^4+\omega^4-2\omega^2\omega_{LO}^2\right)=$$

$$K^2\left(\omega_{TO}^4+\omega^4-2\omega^2\omega_{TO}^2\right)-\frac{\omega^2}{c^2}\varepsilon_\infty\left(\omega_{LO}^2-\omega^2\right)\left(\omega_{TO}^2-\omega^2\right)$$

We now regroup the different powers of ω:

$$-\frac{\omega^6}{c^2}\left(\varepsilon_\infty^2-\varepsilon_\infty\right)+\omega^4\left(K^2\varepsilon_\infty^2+2\frac{\omega_{LO}^2\varepsilon_\infty^2}{c^2}-K^2-\frac{\varepsilon_\infty}{c^2}\left(\omega_{TO}^2+\omega_{LO}^2\right)\right)$$

$$+\omega^2\left(-2\omega_{LO}^2K^2\varepsilon_\infty^2-\omega_{LO}^4\frac{\varepsilon_\infty^2}{c^2}+2\omega_{TO}^2K^2+\frac{\omega_{LO}^2\omega_{TO}^2\varepsilon_\infty}{c^2}\right)$$

$$+K^2\varepsilon_\infty^2\omega_{LO}^4-K^2\omega_{TO}^4=0$$

We now replace $\omega^2=\ell$ and solve the cubic equation that we obtain. Obviously we obtain 3 solutions but we have to pick only the ones with $\omega_{TO}<\omega<\omega_{LO}$.

The 3 solutions for ω^2 are:

$$\rightarrow\ \frac{\varepsilon_\infty\omega_{LO}^2-\omega_{TO}^2}{\varepsilon-1}$$

$$\rightarrow\ -\frac{1}{2\varepsilon_\infty}\Big[-c^2K^2-c^2K^2\varepsilon_\infty-\varepsilon_\infty\omega_{LO}^2-\sqrt{\left(c^2K^2+c^2K^2\varepsilon_\infty+\varepsilon_\infty\omega_{LO}^2\right)^2+4\varepsilon_\infty\left(-c^2K^2\varepsilon_\infty\omega_{LO}^2-c^2K^2\omega_{TO}^2\right)}\Big]$$

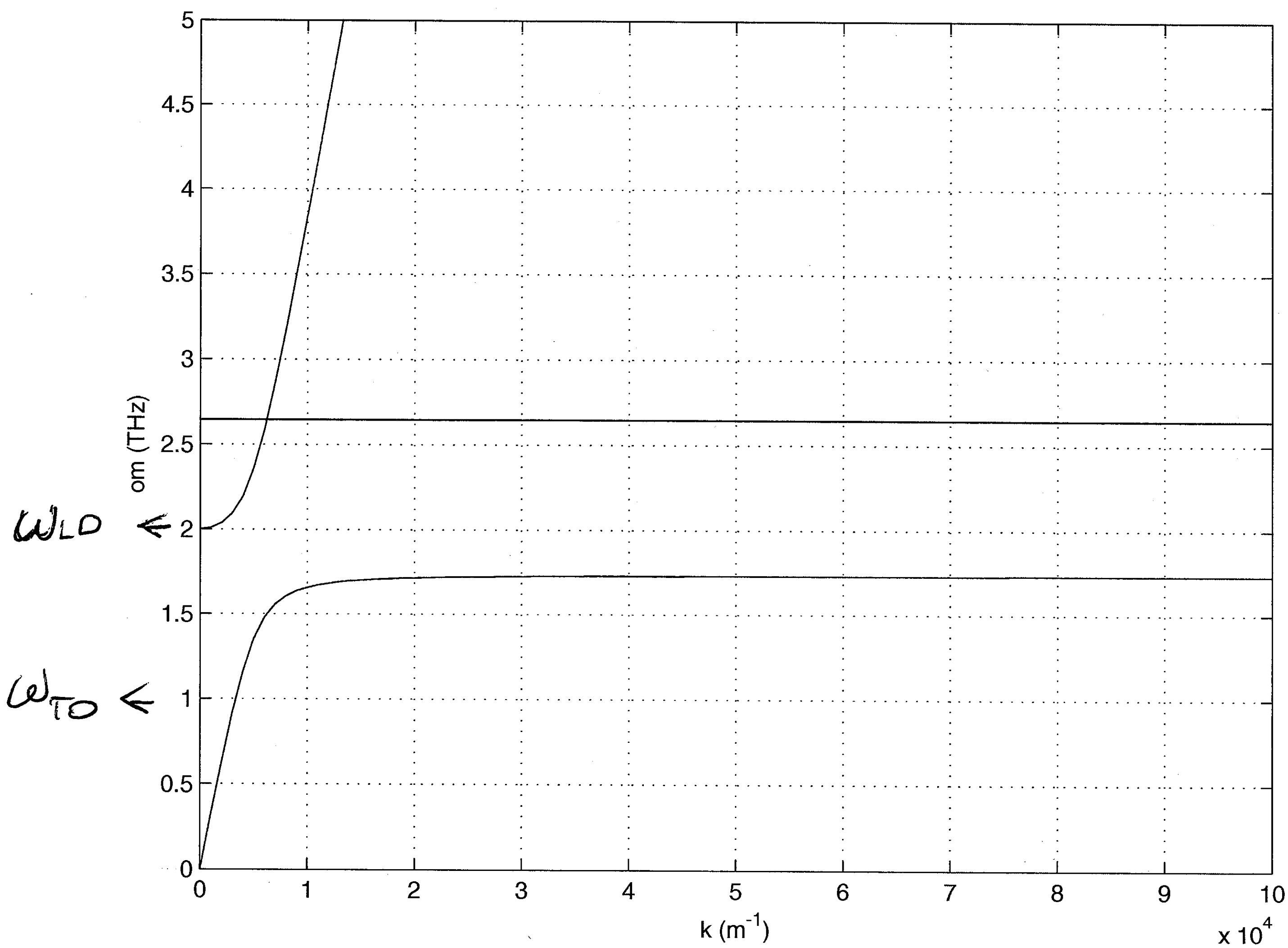
$$\rightarrow\ -\frac{1}{2\varepsilon_\infty}\Big[-c^2K^2-c^2K^2\varepsilon_\infty-\varepsilon_\infty\omega_{LO}^2+\sqrt{\left(c^2K^2+c^2K^2\varepsilon_\infty+\varepsilon_\infty\omega_{LO}^2\right)^2+4\varepsilon_\infty\left(-c^2K^2\varepsilon_\infty\omega_{LO}^2-c^2K^2\omega_{TO}^2\right)}\Big]$$

Once we plot we see that only the third one satisfies the require conditions. ③

To verify that we are doing everything right we check that when $k \to \infty$ $\omega \to \bar{\omega}$ so that $\varepsilon(\bar{\omega}) = -1$

In our case $\varepsilon_\infty \dfrac{\omega_{LO}^2 - \omega^2}{\omega_{TO}^2 - \omega^2} = -1$ gives $\bar{\omega} = \sqrt{\dfrac{\varepsilon_\infty \omega_{LO}^2 + \omega_{TO}^2}{1 + \varepsilon_\infty}}$

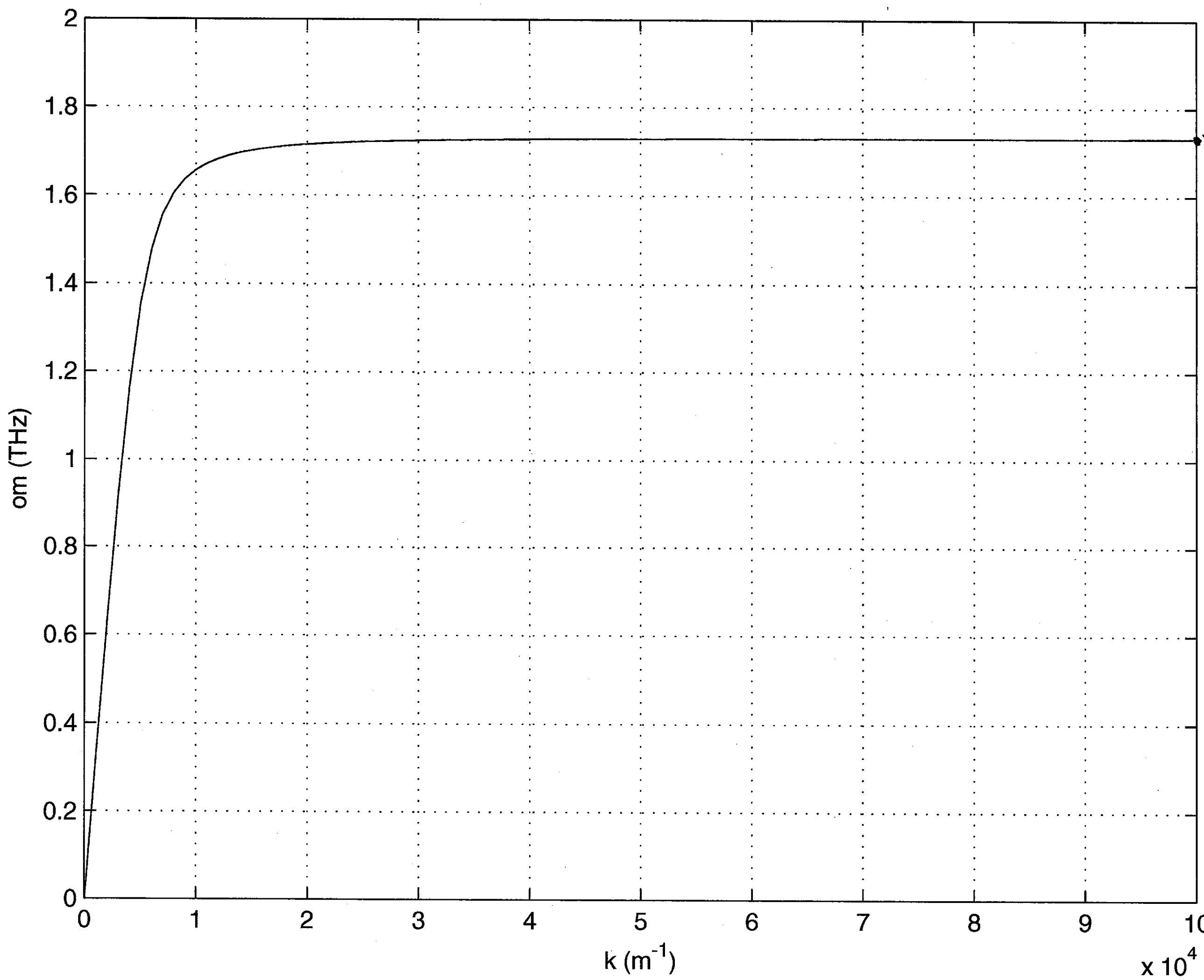
This is actually true.



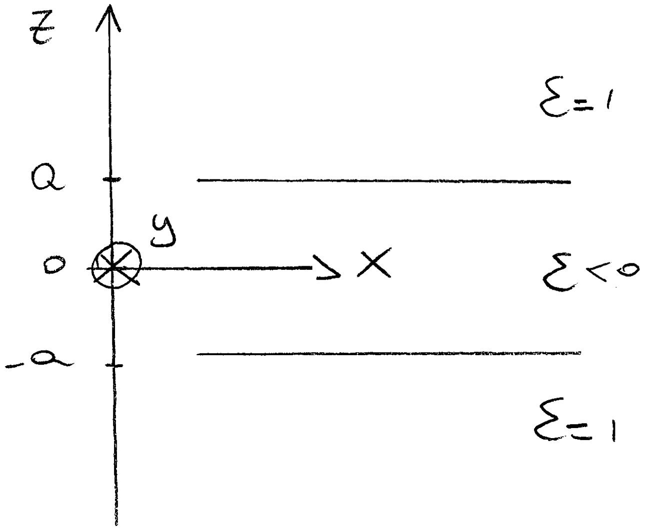
5
4.5
4
3.5
3
2.5
2
1.5
1
0.5
0
om (THz)
ω_{LO}
ω_{TO}
0
1
2
3
4
5
6
7
8
9
10
k (m^{-1})
x 10^4

$$\sqrt{\frac{\varepsilon_\infty \omega_{LO}^2 + \omega_{TO}^2}{1+\varepsilon_\infty}}$$

$\varepsilon_\infty = 2$



PROBLEM 5



PLASMON MODES

We setup the equation for electric and magnetic field in the 3 regions assuming that H exponentially decays away from the $\varepsilon<0$ strip and using $\vec{\nabla}\times\vec{H} = \frac{\varepsilon_r}{c}\frac{\partial \vec{E}}{\partial t}$ to calculate the electric field.

$z > a$

$$\left\{\begin{array}{l} H_y = A \\ E_x = iA\frac{\varkappa c}{\omega} \\ E_z = -\frac{Aqc}{\omega} \end{array}\right. \times e^{iqx-\varkappa z}$$

$$q^2 = k_0^2 + \varkappa^2$$

$z < -a$

$$\left\{\begin{array}{l} H_y = B \\ E_x = -i\frac{B\varkappa c}{\omega} \\ E_z = -\frac{Bqc}{\omega} \end{array}\right. \times e^{iqx+\varkappa z}$$

$|z| < a$ $\qquad q^2 = \varepsilon K_0 + \beta^2$

$$\begin{cases} H_y = C e^{iqx} e^{\beta z} + D e^{iqx} e^{-\beta z} \\ E_x = -i \dfrac{C\beta c}{\omega\varepsilon} e^{iqx} e^{\beta z} + i \dfrac{D\beta c}{\omega\varepsilon} e^{iqx} e^{-\beta z} \\ E_z = -C \dfrac{qc}{\omega\varepsilon} e^{iqx} e^{\beta z} - D \dfrac{qc}{\omega\varepsilon} e^{iqx} \cdot e^{-\beta z} \end{cases}$$

Now we impose the continuity of Hy and Ex in the planes $z = \pm a$:

$$\begin{cases} A e^{-\lambda a} = C e^{\beta a} + D e^{-\beta a} \\ B e^{-\lambda a} = C e^{-\beta a} + D e^{\beta a} \\ \dfrac{iA\lambda c}{\omega} e^{-\lambda a} = -i \dfrac{C\beta c}{\omega\varepsilon} e^{\beta a} + i \dfrac{D\beta c}{\omega\varepsilon} e^{-\beta a} \\ -i \dfrac{B\lambda c}{\omega} e^{-\lambda a} = -i \dfrac{C\beta c}{\omega\varepsilon} e^{-\beta a} + i \dfrac{D\beta c}{\omega\varepsilon} e^{\beta a} \end{cases}$$

Substituting A and B from the first 2 equations we have:

$$\lambda \left(C e^{\beta a} + D e^{-\beta a} \right) = -\frac{C\beta}{\varepsilon} e^{\beta a} + \frac{D\beta}{\varepsilon} e^{-\beta a}$$

$$-\lambda \left(C e^{-\beta a} + D e^{\beta a} \right) = -\frac{C\beta}{\varepsilon} e^{-\beta a} + \frac{D\beta}{\varepsilon} e^{\beta a}$$

Rearranging the terms:

$$C \left(\lambda e^{\beta a} + \frac{\beta}{\varepsilon} e^{+\beta a} \right) + D \left(\lambda e^{-\beta a} - \frac{\beta}{\varepsilon} e^{-\beta a} \right) = 0$$

$$C \left(-\lambda e^{-\beta a} + \frac{\beta}{\varepsilon} e^{-\beta a} \right) + D \left(-\lambda e^{+\beta a} - \frac{\beta}{\varepsilon} e^{\beta a} \right) = 0$$

The system has a non trivial solution only if $\det M = 0$ (5)

$$\begin{vmatrix} 2e^{\beta a} + \beta/\xi \, e^{\beta a} & 2e^{-\beta a} - \beta/\xi \, e^{-\beta a} \\ -2e^{-\beta a} + \beta/\xi \, e^{-\beta a} & -2e^{\beta a} - \beta/\xi \, e^{\beta a} \end{vmatrix} = 0$$

$$-\left(2e^{\beta a} + \beta/\xi \, e^{\beta a}\right)^2 + \left(2e^{-\beta a} - \beta/\xi \, e^{-\beta a}\right)^2 = 0$$

There are 2 possible solutions:

$$\rightarrow 2e^{\beta a} + \beta/\xi \, e^{\beta a} = 2e^{-\beta a} - \beta/\xi \, e^{-\beta a}$$

$$2\sinh(\beta a) = -\beta/\xi \cosh(\beta a)$$

$$\tanh(\beta a) = -\beta/2 \, \frac{1}{\xi}$$

$$\rightarrow 2e^{\beta a} + \beta/\xi \, e^{\beta a} = -2e^{-\beta a} + \beta/\xi \, e^{-\beta a}$$

$$2\cosh(\beta a) = -\beta/\xi \sinh(\beta a)$$

$$\tanh(\beta a) = -\xi \, 2/\beta$$

We can use the definition of 2 and β to get:

$$\tanh\left(\sqrt{q^2-\varepsilon k_0^2}\, a\right) \begin{cases} -\dfrac{1}{\varepsilon}\sqrt{\dfrac{q^2-\varepsilon k_0^2}{q^2-k_0^2}} \\[2ex] -\varepsilon\sqrt{\dfrac{q^2-k_0^2}{q^2-\varepsilon k_0^2}} \end{cases}$$

It is obvious that these solutions are meaningful only if $\varepsilon<0$.

ELECTROSTATIC APPROXIMATION (plasmon)

We already know that in the electrostatic case we have to solve with the appropriate boundary conditions Laplace equation:

$$\nabla^2\phi=0$$

We try a solution that resembles the one already used for the more general electromagnetic case:

$z\leq -a$

$$\phi = A e^{qz}\cdot e^{iqx}$$

$|z|<a$

$$\phi=\left(B e^{-qz}+C e^{qz}\right)e^{iqx}$$

$z>a$

$$\phi = D e^{-qz}\cdot e^{iqx}$$

Now we set again the conditions on the continuity of ϕ and D_z at the z interfaces:

$$A e^{-qa} = B e^{qa} + C e^{-qa}$$

$$D e^{-qa} = B e^{-qa} + C e^{qa}$$

$$A q e^{-qa} = q\left(-B e^{qa} + C e^{-qa}\right)\varepsilon$$

$$D (-q) e^{-qa} = q\left(-B e^{-qa} + C e^{qa}\right)\varepsilon$$

Replacing A and D from the first two equations:

$$B e^{qa} + C e^{-qa} = -B\varepsilon e^{qa} + C\varepsilon e^{-qa}$$

$$-B e^{-qa} - C e^{qa} = -B\varepsilon e^{-qa} + C\varepsilon e^{qa}$$

Rearranging the terms:

$$B e^{qa}(1+\varepsilon) + C e^{-qa}(1-\varepsilon) = 0$$

$$B e^{-qa}(\varepsilon-1) + C e^{qa}(-\varepsilon-1) = 0$$

We again set $\det M = 0$

$$\begin{vmatrix} e^{qa}(1+\varepsilon) & e^{-qa}(1-\varepsilon) \\ e^{-qa}(\varepsilon-1) & e^{qa}(-\varepsilon-1) \end{vmatrix} = 0$$

$$-e^{2qa}(\varepsilon+1)^2 + e^{-2qa}(\varepsilon-1)^2 = 0$$

There are 2 possible solutions:

$$e^{qa}(\varepsilon+1) = e^{-qa}(\varepsilon-1)$$

which gives $\tanh(qa) = -\frac{1}{\varepsilon}$

and

$$e^{qa}(\varepsilon+1) = e^{-qa}(1-\varepsilon)$$

which leads to $\tanh(qa) = -\varepsilon$

In summary:

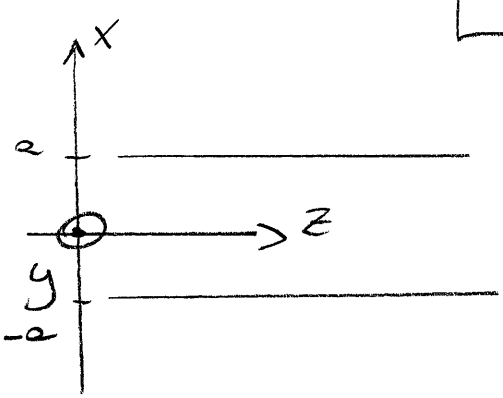
$$\tanh(qa) \begin{cases} -\varepsilon \\ -\frac{1}{\varepsilon} \end{cases}$$

This is the limit for $q \gg \sqrt{\varepsilon}\, k_0$ of the electromagnetic solution.

WAVEGUIDE MODES

We have to distinguish the TE from the TM case.

$\boxed{\text{TE}}$



$$\vec{E} = E\hat{y}$$

We consider again a solution in the 3 different regions: (7)

$x > a$

$$E = E_1 e^{-\gamma x} \qquad \gamma = \sqrt{\beta^2 - k_0^2} > 0$$

$|x| < a$

$$E = E_s \cos(hx) + E_a \sin(hx) \qquad h = \sqrt{\varepsilon k_0^2 - \beta^2} > 0$$

$x < -a$

$$E = E_3 e^{\gamma x}$$

We consider separately the symmetric and antisymmetric case.

The boundary conditions are set only at $x = a$ and $x = -a$

$|E_3| = |E_1|$.

Since $\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$ we have

$$\vec{\nabla} \times \vec{E} = -i\omega\mu\vec{H}$$

So $-\frac{\partial E_y}{\partial z}\hat{x} + \frac{\partial E_y}{\partial x}\hat{z} = -i\omega\mu\vec{H}$

This means that we have to require the continuity of E_y and $\frac{\partial E_y}{\partial x}$.

$$E_1 e^{-\gamma a} = E_s \cos(ha)$$

$$-\gamma E_1 e^{-\gamma a} = -h E_s \sin(ha)$$

$$\boxed{\operatorname{tg}(ha) = \gamma/h}$$

For the antisymmetric mode:

$$E_1 e^{-\gamma a} = E_a \sin(ha)$$

$$-\gamma E_1 e^{-\gamma a} = h E_a \cos(ha)$$

$$\boxed{ctg(ha) = -\gamma/h}$$

In summary:

Symmetric: $$tg\left(\sqrt{\varepsilon k_0^2 - \beta^2}\, a\right) = \sqrt{\frac{\beta^2 - k_0^2}{\varepsilon k_0^2 - \beta^2}}$$

Antisymmetric: $$ctg\left(\sqrt{\varepsilon k_0^2 - \beta^2}\, a\right) = -\sqrt{\frac{\beta^2 - k_0^2}{\varepsilon k_0^2 - \beta^2}}$$

TM

$$\vec{H} = H\hat{y}$$

We repeat the same procedure:

$x > a$

$$H = H_1 e^{-\gamma x} \qquad \gamma = \sqrt{\beta^2 - k_0^2}$$

$|x| < a$

$$H = H_s \cos(hx) + H_a \sin(hx) \qquad h = \sqrt{\varepsilon k_0^2 - \beta^2}$$

$x < -a$

$$H = H_3 e^{\gamma x}$$

We can use $\vec{\nabla}\times\vec{H}=\frac{\partial\vec{D}}{\partial t}$

$$-\frac{\partial H_y}{\partial z}\hat{x}+\frac{\partial H_y}{\partial x}\hat{z}=i\omega\varepsilon\vec{E}$$

We have thus to require the continuity of H_y and $\frac{\partial H_y}{\partial x}\cdot\frac{1}{\varepsilon}$

symmetric mode

$$\left.\begin{aligned} H_1 e^{-\gamma a} &= H_s\cos(ha) \\ -\gamma H_1 e^{-\gamma a} &= -\frac{H_s h}{\varepsilon}\sin(ha)\end{aligned}\right\} \Rightarrow \boxed{tg(ha)=\frac{\gamma}{h}\varepsilon}$$

antisymmetric mode

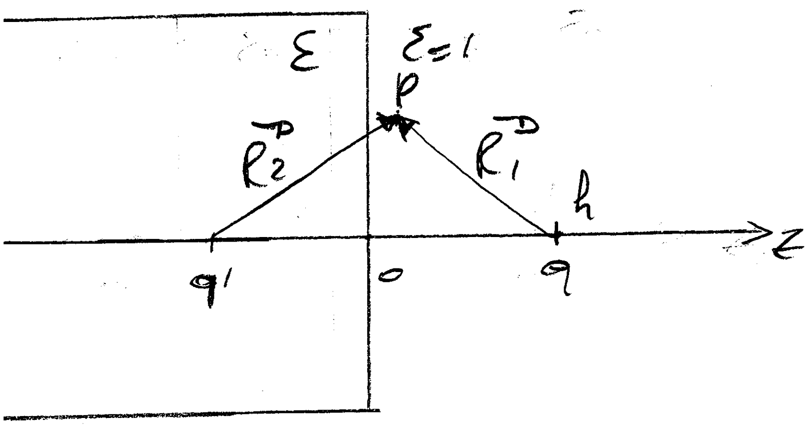
$$\left.\begin{aligned} H_1 e^{-\gamma a} &= H_a\sin(ha) \\ -\gamma H_1 e^{-\gamma a} &= \frac{H_a\cdot h}{\varepsilon}\cos(ha)\end{aligned}\right\} \Rightarrow \boxed{ctg(ha)=-\frac{\gamma}{h}\varepsilon}$$

In summary:

Symmetric: $tg\left(\sqrt{\varepsilon k_0^2-\beta^2}\,a\right)=\varepsilon\sqrt{\frac{\beta^2-k_0^2}{\varepsilon k_0^2-\beta^2}}$

Antisymmetric: $ctg\left(\sqrt{\varepsilon k_0^2-\beta^2}\,a\right)=-\varepsilon\sqrt{\frac{\beta^2-k_0^2}{\varepsilon k_0^2-\beta^2}}$

METHOD OF IMAGES (2 semispaces)



When $z>0$

$$\phi = \frac{1}{4\pi}\left(q/R_1 + q'/R_2\right)$$

When $z<0$

$$\phi = \frac{1}{4\pi\varepsilon}\frac{q''}{R_1}$$

This is the simplest hypothesis we can make at this point. We require $E_{tangential}$ and D_{normal} to be continuous at $z=0$.

$$\frac{\partial}{\partial z}\left(\frac{1}{R_1}\right)\Big|_{z=0} = -\frac{\partial}{\partial z}\left(\frac{1}{R_2}\right)\Big|_{z=0} = \frac{d}{(\rho^2+h^2)^{3/2}}$$

$$\frac{\partial}{\partial \rho}\left(\frac{1}{R_1}\right)\Big|_{z=0} = \frac{\partial}{\partial \rho}\left(\frac{1}{R_2}\right)\Big|_{z=0} = \frac{-\rho}{(\rho^2+h^2)^{3/2}}$$

We thus get

$$\begin{cases} q - q' = q'' \\ q + q' = q''/\varepsilon \end{cases}$$

this gives:

$$q' = -\frac{\varepsilon-1}{\varepsilon+1}q$$

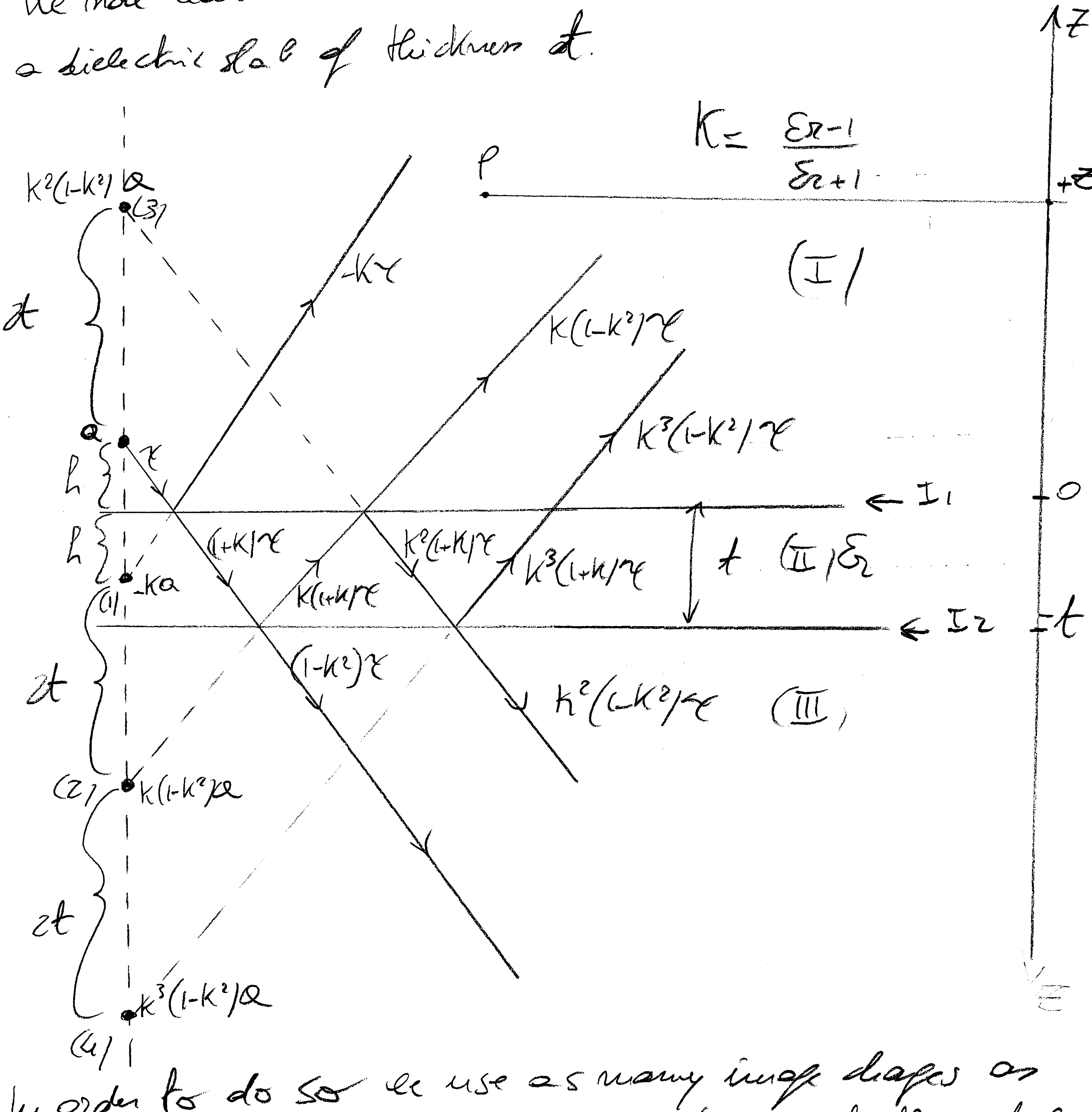
$$q'' = \frac{2\varepsilon}{\varepsilon+1}q$$

So we stay in the $z>0$ region the potential at a point P is

$$\phi = \frac{q}{4\pi R_1} + \frac{1-\varepsilon}{1+\varepsilon}\frac{q}{4\pi R_2}$$

It is obvious from this last equation that when $\varepsilon_r \to -1$ (11)
the potential and thus the field diverges.

We now want to solve the same problem when we have a dielectric slab of thickness t.



In order to do so we use as many image charges as needed to satisfy the boundary conditions at the interfaces of the dielectric slab

At I_1 we need a charge (1) at distance h from I_1 whose value is $-KQ$. At I_2 we see a charge $Q(1+K)$ sitting at the place of the original charge Q so we need another image charge Q_2 whose value is $K(1+K)$ and which is located at $h+t$ from I_2. This charge though is seen as a $K(1-K^2)Q$ charge in region (I).

So at this point outside the slab we have the charge Q, $-KQ$, $K(1-K^2)Q$ in the positions indicated in figure. It's easy to continue this analysis

In conclusion at P we have

$$\phi = \frac{Q}{4\pi\varepsilon_0}\frac{1}{\left((z_0-h)^2+\rho^2\right)^{1/2}} + \frac{Q}{4\pi\varepsilon_0}(1-K^2)\sum_{n=1}^{+\infty} K^{2n-1}\frac{1}{\left(\rho^2+(z_0+h+2nt)^2\right)^{1/2}}$$

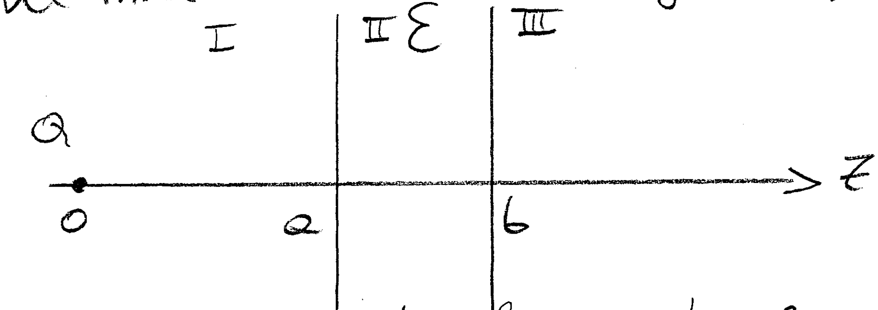
$$-\frac{QK}{4\pi\varepsilon_0\left((z_0+h)^2+\rho^2\right)^{1/2}} =$$

There is actually another approach to solve this problem that relies on one property of the Bessel functions:

FROM "Static and dynamic electricity" SMYTHE

$$\frac{q}{4\pi\varepsilon r} = \frac{q}{4\pi\varepsilon}\int_0^{+\infty} J_0(k\rho)\, e^{-kz}\, dk$$

We may use this for the following geometry:



We have again to follow Laplace's eqn; obviously $\frac{Q}{4\pi\varepsilon r}$ satisfies it and if we add any function of k under the integral we'll still have a solution.

So we guess the solution in the 3 different regions:

$$V_1 = \frac{Q}{4\pi\varepsilon_0}\left[\int_0^{+\infty} J_0(k\rho)\, e^{-k|z|}\, dk + \int_0^{+\infty} \phi(k)\, J_0(k\rho)\, e^{kz}\, dk\right]$$

$$V_2 = \frac{q}{4\pi\varepsilon_0}\left[\int_0^{+\infty} \Psi(k)\, J_0(k\rho)\, e^{-kz}\, dk + \int_0^{+\infty} \theta(k)\, J_0(k\rho)\, e^{kz}\, dk\right]$$

$$V_3 = \frac{q}{4\pi\varepsilon_0}\left[\int_0^{+\infty} \Omega(k)\, J_0(k\rho)\, e^{-kz}\, dk\right]$$

Using the Fourier Bessel integral we can show that if $\int_0^{+\infty} f_1(K) J_0(K\rho)\, dK = \int_0^{+\infty} f_2(K) J_0(K\rho)\, dK$ then

$$f_1(x) = f_2(x)$$

and applying the boundary conditions $V_1 = V_2$ and $\frac{\partial V_1}{\partial z} = \frac{\partial V_2}{\partial z}$ at $z = a$ and $V_2 = V_3$ and $\frac{\partial V_2}{\partial z} = \frac{\partial V_3}{\partial z}$ at $z = b$ we get:

$$e^{-Ka} + \phi(K) e^{Ka} - \Psi(K) e^{-Ka} - \Theta(K) e^{Ka} = 0$$

$$-e^{-Ka} + \phi(K) e^{Ka} + \varepsilon \Psi(K) e^{-Ka} - \varepsilon \Theta(K) e^{Ka} = 0$$

$$\Psi(K) e^{-Kb} + \Theta(K) e^{Kb} - \Omega(K) e^{-Kb} = 0$$

$$-\varepsilon \Psi(K) e^{-Kb} + \varepsilon \Theta(K) e^{Kb} + \Omega(K) e^{-Kb} = 0$$

We'll only find $\Omega(K)$ so that we can see if V_3 diverges: if this is the case also V_1 and V_2 will diverge.

$$\Omega(K) = \frac{4\varepsilon}{(\varepsilon+1)^2 - (\varepsilon-1)\, e^{2(a-b)K}}$$

$$V_3 = \frac{Q}{4\pi\varepsilon_0} \int_0^{+\infty} \frac{4\varepsilon}{(\varepsilon+1)^2 - (\varepsilon-1)\, e^{2K(a-b)}} J_0(K\rho) e^{-Kz}\, dK$$

We must replace $\mathcal{E}$ with -1:

$$V_3 = \frac{Q}{4\pi\varepsilon_0}\int_0^{+\infty} J_0(k\rho)\, e^{k(2(b-a)-z)}\, dk$$

The integral diverges since $0 \leq k \leq +\infty$ unless:

$$2(b-a) - z < 0$$

$$z > 2(b-a)$$

Since $z \geq b$ we have that if

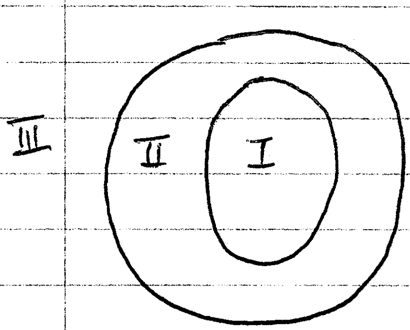
$$b < 2(b-a) \quad \text{i.e.} \quad b > 2a$$

we'll have divergence for some values of z.

In summary the potential will diverge if $\varepsilon = -1$ and $a < b/2$ or $a < (b-a)$. This means that Q has to be sufficiently close to the slab to excite the surface plasmon modes.

Problem 6

(1) ELECTROMAGNETIC CASE



For source free electromagnetic field with time dependent $e^{-i\omega t}$

$$(\nabla^2+\varepsilon k_0^2)\vec{H}=0, \quad \varepsilon=1 \text{ for region I, III.} \quad k_0=\omega/c$$

To find plasmon resonance, we only consider TM mode. And assume the field are independent of ϕ. Due to $\nabla\cdot\vec{B}=0$, the only non-zero magnetic field component is H_ϕ.

In spherical coordinate

$$\frac{1}{r^2}\partial_r(r^2\partial_r H_\phi)+\frac{1}{r^2\sin\theta}\partial_\theta(\sin\theta\,\partial_\theta H_\phi)+\varepsilon k^2 H_\phi=0$$

Write $\quad H_\phi=f_\ell(r)P_\ell(\cos\theta)=\frac{1}{r^{1/2}}U_\ell(r)\,P_\ell(\cos\theta)$

Plug into above equation, we get

$$\left[\frac{d^2}{dr^2}+\frac{1}{r}\frac{d}{dr}+\varepsilon k_0^2-\frac{\nu^2}{r^2}\right]U_\ell(r)=0$$

with $k_0=\omega/c$, $\nu=\ell+\frac{1}{2}$

This is the Bessel's differential equation, and two linearly independent solutions are $J_{\ell+\frac{1}{2}}(\sqrt{\varepsilon}k_0 r)$, $N_{\ell+\frac{1}{2}}(\sqrt{\varepsilon}k_0 r)$

Note $\quad z_\ell(x) = \sqrt{\frac{\pi}{2x}}\, Z_{\ell+\frac{1}{2}}(x) \quad \left(\begin{array}{l} z_\ell = j_\ell, n_\ell \\ Z_\ell = J_\ell, N_\ell \end{array} \right)$

$\therefore \quad f_\ell(r) = A_\ell j_\ell(\sqrt{\varepsilon} k_0 r) + B_\ell n_\ell(\sqrt{\varepsilon} k_0 r)$

Consider the boundary condition at $r=0$, & $r \to \infty$

$r=0$: $f_\ell(r)$ finite

$r \to \infty$: $f_\ell^{(r)} \propto e^{ik_0 r}/r$

And to have plasmon mode, $\varepsilon < 0$

then.

In region I:

$$f_\ell(r) = A_\ell j_\ell(k_0 r)$$

In region II:

$$f_\ell(r) = B_\ell j_\ell(ikr) + C_\ell n_\ell(ikr), \quad k = \sqrt{|\varepsilon|}\, k_0$$

In region III:

$$f_\ell(r) = D_\ell h_\ell^{(1)}(k_0 r)$$

Note $\quad E_\theta = \frac{1}{i\omega\varepsilon} \frac{1}{r} \partial_r (r H_\phi)$

Therefore.

In region I:

$H_\phi = A_\ell j_\ell(k_0 r) P_\ell(\cos\theta)$

$$E_\theta = \frac{A_\ell}{i\omega}\left[\frac{1}{r} j_\ell(k_0 r) + j_\ell'(k_0 r)\right] P_\ell(\cos\theta)$$

In region II:

$$H_\phi = [B_\ell j_\ell(ikr) + C_\ell n_\ell(ikr)]\, P_\ell(\cos\theta)$$

$$E_\theta = \frac{1}{i\omega\varepsilon}\left\{\left[\frac{1}{r} j_\ell(ikr) + j_\ell'(ikr)\right]B_\ell + \left[\frac{1}{r} n_\ell(ikr) + n_\ell'(ikr)\right]C_\ell\right\} P_\ell(\cos\theta)$$

In region III:

$$H_\phi = D_\ell h_\ell^{(1)}(k_0 r)\, P_\ell(\cos\theta)$$

$$E_\theta = \frac{1}{i\omega} D_\ell \left[\frac{1}{r} h_\ell^{(1)}(k_0 r) + h_\ell^{(1)\prime}(k_0 r)\right] P_\ell(\cos\theta)$$

where $z_\ell'(kr) = \frac{d}{dr} z_\ell(kr)$ $\quad (z_\ell = j_\ell, n_\ell, h_\ell^{(1)})$.

The continuity of H_ϕ & E_θ at $r = r_1$ gives

$$\begin{cases} A_\ell j_\ell(k_0 r_1) = B_\ell j_\ell(ikr_1) + C_\ell n_\ell(ikr_1) \\ \varepsilon A_\ell [j_\ell(k_0 r_1) + r_1 j_\ell'(k_0 r_1)] = [j_\ell(ikr_1) + r_1 j_\ell'(ikr_1)] B_\ell \\ \qquad + [n_\ell(ikr_1) + r_1 n_\ell'(ikr_1)] C_\ell \end{cases}$$

$$\rightarrow \quad \alpha_\ell^{(1)} B_\ell + \beta_\ell^{(1)} C_\ell = 0$$

where

$$\begin{bmatrix} \alpha_\ell^{(1)} \\ \beta_\ell^{(1)} \end{bmatrix} = \varepsilon \begin{bmatrix} j_\ell(ikr_1) \\ n_\ell(ikr_1) \end{bmatrix} [j_\ell(k_0 r_1) + r_1 j_\ell'(k_0 r_1)] - j_\ell(k_0 r_1)\left\{\begin{bmatrix} j_\ell(ikr_1) \\ n_\ell(ikr_1) \end{bmatrix} + r_1 \begin{bmatrix} j_\ell'(ikr_1) \\ n_\ell'(ikr_1) \end{bmatrix}\right\}$$

The continuity of H_ϕ, E_θ at $r=r_2$ gives

$$\begin{cases} D_\ell h_\ell^{(1)}(k_0 r_2) = B_\ell j_\ell(ikr_2) + C_\ell n_\ell(ikr_2) \\ \varepsilon D_\ell \left(h_\ell^{(1)}(k_0 r_2) + r_2 h_\ell^{(1)\prime}(k_0 r_2)\right) = \left[j_\ell(ikr_2) + r_2 j_\ell'(ikr_2)\right] B_\ell \\ \qquad + \left[n_\ell(ikr_2) + r_2 n_\ell'(ikr_2)\right] C_\ell . \end{cases}$$

$$\rightarrow \quad \alpha_\ell^{(2)} B_\ell + \beta_\ell^{(2)} C_\ell = 0$$

where
$$\begin{bmatrix} \alpha_\ell^{(2)} \\ \beta_\ell^{(2)} \end{bmatrix} = \varepsilon \begin{bmatrix} j_\ell(ikr_2) \\ n_\ell(ikr_2) \end{bmatrix} \left[h_\ell^{(1)}(k_0 r_2) + r_2 h_\ell^{(1)\prime}(k_0 r_2)\right] - h_\ell^{(1)}(k_0 r_2) \left\{ \begin{bmatrix} j_\ell(ikr_2) \\ n_\ell(ikr_2) \end{bmatrix} + r_2 \begin{bmatrix} j_\ell'(ikr_2) \\ n_\ell'(ikr_2) \end{bmatrix} \right\}$$

To have non-zero B_ℓ, C_ℓ, it requires

$$\frac{\alpha_\ell^{(1)}}{\alpha_\ell^{(2)}} = \frac{\beta_\ell^{(1)}}{\beta_\ell^{(2)}}$$

From the above equation, we can solve for ε_ℓ at which there is a plasmon resonance.

In electrostatic limit,

$kr_1, kr_2, k_0r_1, k_0r_2 \ll 1$

Recall $j_\ell(x) \simeq \frac{x^\ell}{(2\ell+1)!!}$ when $x \ll 1$

$n_\ell(x) \simeq -\frac{(2\ell-1)!!}{x^{\ell+1}}$

we get

$$\alpha_\ell^{(1)} = \frac{(\varepsilon-1) i^\ell |\varepsilon|^{\ell/2} (k_0r_1)^{2\ell}}{[(2\ell+1)!!]^2} (\ell+1)$$

$$\beta_\ell^{(1)} = -\frac{\ell+(\ell+1)\varepsilon}{i^{\ell+1}|\varepsilon|^{\frac{\ell+1}{2}}(2\ell+1)(k_0r_1)}$$

$$\alpha_\ell^{(2)} = \frac{i^{\ell+1}|\varepsilon|^{\ell/2}}{(2\ell+1)(k_0r_2)}[\varepsilon\ell+(\ell+1)]$$

$$\beta_\ell^{(2)} = \frac{[(2\ell-1)!!]^2}{i^\ell|\varepsilon|^{\frac{\ell+1}{2}}(k_0r_2)^{2\ell+2}}\ell(1-\varepsilon)$$

plug them into $\frac{\alpha_\ell^{(1)}}{\alpha_\ell^{(2)}} = \frac{\beta_\ell^{(1)}}{\beta_\ell^{(2)}}$, with some

algebra, we get.

$$\boxed{(1-\beta^{2\ell+1})\varepsilon^2+\left(\frac{\ell}{\ell+1}+\frac{\ell+1}{\ell}+2\beta^{2\ell+1}\right)+(1-\beta^{2\ell+1})=0}$$

Where $\beta = r_1/r_2$

(2) ELECTROSTATIC CASE

The equation we are solving is

$$\nabla^2\Phi=0$$

In spherical coordinate

$$\frac{1}{r}\partial_r^2(r\Phi)+\frac{1}{r^2\sin\theta}\partial_\theta(\sin\theta\,\partial_\theta\Phi)+\frac{1}{r^2\sin^2\theta}\partial_\phi^2\Phi=0$$

The general solution is (Jackson P110)

$$\Phi=\sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell}\left[A_{\ell m}r^{\ell}+B_{\ell m}/r^{\ell+1}\right]Y_{\ell m}(\theta,\phi)$$

Considering the boundary condition

$$\begin{cases}\Phi\to 0, & \text{when } r\to\infty\\ \Phi\ \text{finite}, & \text{when } r\to 0\end{cases}$$

Then we have

In region I: $\Phi_I=\sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell}A_{\ell m}^{(I)}r^{\ell}Y_{\ell m}(\theta,\phi)$

In region II: $\Phi_{II}=\sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell}\left(A_{\ell m}^{(II)}r^{\ell}+B_{\ell m}^{(II)}/r^{\ell+1}\right)Y_{\ell m}(\theta,\phi)$

In region III: $\Phi_{III}=\sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell}B_{\ell m}^{(III)}/r^{\ell+1}\,Y_{\ell m}(\theta,\phi)$

The continuity of Φ & D_r at $r=r_1$ gives

$$\begin{cases} A^{(I)}_{lm} = A^{(II)}_{lm} + B^{(II)}_{lm} / r_1^{2l+1} \\ A^{(I)}_{lm} = \varepsilon A^{(II)}_{lm} - \frac{\varepsilon(l+1)}{l} B^{(II)}_{lm} / r_1^{2l+1} \end{cases} \quad \text{for } l \geq 1.$$

$$\rightarrow \quad (-1+\varepsilon) A^{(II)}_{lm} = \frac{\varepsilon(l+1)+l}{l} \frac{1}{r_1^{2l+1}} B^{(II)}_{lm} = 0.$$

The continuity of Φ & D_r at $r=r_2$ gives

$$\begin{cases} A^{(II)}_{lm} r_2^{2l+1} + B^{(II)}_{lm} = B^{(III)}_{lm} \\ \frac{\varepsilon l}{l+1} A^{(II)}_{lm} r_2^{2l+1} - \varepsilon B^{(II)}_{lm} = -B^{(III)}_{lm} \end{cases} \quad \text{for } l \geq 1$$

$$\rightarrow \quad \frac{(\varepsilon+1)l+1}{l+1} r_2^{2l+1} A^{(II)}_{lm} = (\varepsilon-1) B^{(II)}_{lm}$$

To have non-zero $A^{(II)}_{lm}$ & $B^{(II)}_{lm}$, it requires

$$\frac{(\varepsilon-1)(l+1)}{[\varepsilon l + (l+1)] r_2^{2l+1}} = \frac{\varepsilon(l+1)+l}{l(\varepsilon-1) r_1^{2l+1}}$$

$$\rightarrow \quad (1-\beta^{2l+1})\varepsilon^2 + \left(\frac{l}{l+1} + \frac{l+1}{l} + 2\beta^{2l+1}\right)\varepsilon + (1-\beta^{2l+1}) = 0$$

where $\beta = r_1/r_2$, $l \geq 1$

$$\left(A^{(I)}_{00} = A^{(II)}_{00} = B^{(II)}_{00} = B^{(III)}_{00} = 0 \right)$$

The result is the same as we get from electromagnetic case, then take the electrostatic limit.

(3) ELECTRIC FIELD DUE TO A PIONT CHARGE AT THE CENTER

Due to the spherical symmeotry

① $l=0$, $m=0$.

By Gaussian's Theorem

② $E_{I} = E_{III} = \frac{Q}{4\pi\varepsilon_0}\frac{1}{r^2}$

$\therefore$ $\Phi_{I} = -\frac{\alpha}{r} + C_1$

$\Phi_{II} = \frac{B}{r} + C_2$ where $\alpha = Q/(4\pi\varepsilon_0)$

$\Phi_{III} = -\frac{\alpha}{r} + C_3$

the continity of ~~E and~~ D_r at $r=r_1$ gives

$$\frac{\alpha}{r_1^2} = -\frac{\varepsilon B}{r_1^2}$$

$\rightarrow$ $B = -\frac{\alpha}{\varepsilon}$

$\therefore$ The electric field is

$E_{I} = E_{III} = \frac{\alpha}{r^2}$ ($r<r_1$, or $r>r_2$)

$E_{II} = \frac{\alpha}{\varepsilon r^2}$ ($r_1<r<r_2$), $\alpha = \frac{Q}{4\pi\varepsilon_0}$

No divergence. indicating that this electric field can not couple to the plasmon mode found in (2).