Electromagnetic screening by metals

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To dispel a widespread but erroneous belief among physicists that the penetration of ac magnetic fields into normal metals is determined by the usual skin depth δ alone, a simple analysis is presented of two problems in each of which a different length scale determines the effective screening. For a cylindrical can of thickness $d \ll \delta$ and radius $R \ll \lambda$, where λ is the wavelength, it is shown that the critical thickness for effective screening is $d_c = \delta^2/R$. For a planar film with thickness $d \ll \delta$, $d_c = c/2\pi\sigma$, where σ is the conductivity. An exact analysis is also'presented of the screening for a cylinder of arbitrary thickness, as well as an analogy between screening by normal metals and screening by superconductors.

It is widely and erroneously believed by physicists that problems of the penetration of alternating magnetic fields into normal metals may be discussed in terms of a single length, the skin depth δ, defined in Gaussian units as

$$\delta = (c^2/2\pi\omega\sigma)^{1/2}, \tag{1}$$

where σ is the electrical conductivity. For brevity, we have taken the permeability as unity and we have also assumed that the carrier mean free path $l \ll 6$, in order to exclude the regime of the anomalous skin effect. Very roughly, $\delta \sim 1$ cm at 60 Hz for copper at room temperature, with $\sigma = 5.4x\ 10^{17}\ \mathrm{s}^{-1}$.

A variety of fundamental lengths describe the shielding and penetration problems of interest in physics. We present here an explication of two fundamental lengths that are very important yet have escaped emphasis in textbooks. The historical skin depth calculation that leads to (1) refers to a circular wire that carries an electromagnetic field along the wire, with wavelength/z $\gg R$, where R is the radius of the wire. In the limit $\delta \ll R$, the field falls off exponentially from the outside of the wire, with a characteristic length given by (1) . This result is derived in several textbooks, in detail by Smythe.' Most textbooks treat only the penetration of an ac magnetic field into a semi-infinite solid bounded by a plane; this is simple mathematically and does not involve the Kelvin functions, but the postulated limit $\lambda \gg R$ is necessarily violated when we consider the plane as the limit of a cylinder. This violation creates difficulties for the conventional problem of the semi-infinite solid.

The-physics is quite different when the magnetic field is applied by a solenoid of circular cross section enclosing a circular cylindrical shell of inner radius R and outer radius $R + d$. The simple argument below, based on the Faraday and Ampere laws, shows that the interior of the tube is effectively screened from the applied field when $d > d_0$, where

$$d_0 = \delta^2/R = \delta(\delta/R), \tag{2}$$

with δ defined by (1) . The applied field may be screened when the can thickness is much less than a skin depth: If $R/\delta = 100$, the screening starts to be effective when $d/\delta = 0.01$! Do not resist this result; for example, it may come into play whenever thin aluminum shielding cans are used in the laboratory to screen 60-Hz disturbances. The Maxwell' moving image theory of eddy currents displays the importance of the specific geometry of an eddy current problem.

If the metal can in this example is made of a superconductor, a very thin can may screen a magnetic field applied by the solenoid. If A is the nominal penetration depth of the London equation, then we can show that the critical wall thickness required for screening is

$$d_{\mathrm{crit}} = \Lambda^2/R. \tag{3}$$

This result was discussed recently,[3] together with the important limitations that fluxoid motion imposes on its applicability.

Consider a third example: A thin metallic film will reflect and absorb a large fraction of the radiation normally incident when the film thickness $d > d_0$, with

$$d_0 = c/2\pi\sigma. \tag{4}$$

Recall that the conductivity σ has the dimensions of frequency in Gaussian units, so that c/σ is just the fundamental length that enters the Maxwell equation, curl $H = 4\pi\sigma E/c + \partial D/c\,\partial t$. In the example given by Tinkham,[4] a 2-nm film of tin transmitted only $\frac{1}{4}$ of the incident radiation, independent of ω at a frequency such that the skin depth $\delta \approx 100$ nm. This is quite a different class of problem in which the skin depth has nothing to do with the response of the system.

I. SCREENING BY A THIN CAN IN A LONGITUDINAL AC MAGNETIC FIELD

We give a direct physical argument for the result (2) above, in the limit $\lambda \gg R$. The magnetic field H_i inside the can of circular cross section is $H_i = H_0 + H_s$, where H_0 is the applied field external to the can and H_s is the screening field produced by the back emf around the circumference of the can. The time dependence is exp $(-i\omega t)$. Let R and $R + d$ be the inner and outer radii of the can. We assume that $R \gg \delta \gg d$ is generally satisfied. Then the back emf is, by the Faraday law,'

$$V = -\frac{1}{c}\frac{\partial\Phi}{\partial t} = \frac{i\omega\pi R^2 H_i}{c}, \tag{5}$$

which creates a current

$$J_\phi = (V/2\pi R)\sigma d = i\omega\sigma R d H_i/2c \tag{6}$$

per unit length of the can. By the Ampere law,

$$H_s = i\frac{4\pi}{c}J_\phi = i\frac{2\pi\omega\sigma Rd}{c^2}H_i = i\frac{Rd}{\delta^2}H_i. \tag{7}$$

The screening factor is given by

$$H_i/H_0 = 1/[1 - i(Rd/\delta^2)] \tag{8}$$

Kin the metal when $|z_i - z_o| \ll 1$, so that our physical argument that leads to (8) applies in the regime $d \ll \delta$, as expected.

What happens when propagation is taken into acount? That is, when the wavelength λ of the radiation is less than the radius of the cylinder? Instead of all the flux in the cylinder being added or removed every quarter cycle, only the flux within a distance λ of the wall of the cylinder is added or removed. In this limit, the rate of change of the flux is proportional to $R\lambda$ instead of R^2, we replace R with λ in the screening result (8). The screening parameter Rd/δ^2 becomes $\lambda d/\delta^2$, which is of the order of $\lambda\omega\sigma d/c^2 \sim da/c$, just as in the thin film transmission example treated in detail below.

IL CRITICAL TRANSMISSION THICKNESS OF A METALLIC FILM

A metallic film can reflect and absorb most of the incident light at a thickness much less than the skin depth δ. Most of the radiation is transmitted when the thickness is less than the critical thickness $d_0 = c/2\pi\sigma$ given by (4) above. For copper at room temperature $\sigma = 5 \times 10^{17}\ \mathrm{s}^{-1}$, so that $d_0 \sim 10^{-8}$ cm; or, with the adjustment to σ referred to in Ref. 4, the thickness will be an order of magnitude greater. The ratio of d_0 to the skin depth δ is $(c/2\pi\sigma)/(c^2/2\pi\omega\sigma)^{1/2} = (\omega/2\pi\sigma)^{1/2} \ll 1$ in the infrared, with $\omega \sim 10^{13}\ \mathrm{s}^{-1}$.

The physical argument for the effect of thin films is that as d increases and approaches d_0, the electric field E in the film creates a current sheet that, by Ampere's law, causes a magnetic field H of the same magnitude as the magnetic field in the incident radiation. The two magnetic fields interfere to cancel the transmitted beam.

The solution of the complete problem as given in standard optics texts can be simplified considerably for the thin films of interest, of thickness much less than the skin depth 6. The standard solution starts by writing, with all coefficients having the time-dependence $\exp(-i\omega t)$, the magnetic field (taken in the z direction) of the incident and reflected waves in the vacuum region 1 for $x \leqslant 0$,

$$H(x) = H_1^+ \exp(ikx) + H_1^- \exp(-ikx); \tag{25}$$

in the film, called region 2, for $0 \leqslant x \leqslant d$,

$$H(x) = H_2^+ \exp(i\kappa x) + H_2^- \exp(-i\kappa x); \tag{26}$$

and for the transmitted wave in vacuum (region 3) for $x \geqslant d$,

$$H(x) = H_3^+ \exp(ikx). \tag{27}$$

The wave equation in the vacuum gives the dispersion relation $\omega = ck$; and in the metal $\omega = n(\omega)c\kappa$, where the complex refractive index is

$$n(w) = (1 + 4\pi i\sigma/\omega)^{1/2} \tag{28}$$

from the usual expression of the first Maxwell equation, curl $H(w) = -i\omega n^2 E(\omega)/c$.

The boundary conditions on the magnetic field components are, at $x = 0$,

$$H_1^+ + H_1^- = H_2^+ + H_2^-; \tag{29}$$

and at $x = d$,

$$H_2^+ \exp(i\kappa d) + H_2^- \exp(-i\kappa d) = H_3^+ \exp(ikd). \tag{30}$$

It simplifies the problem if we consider the physics that led to these boundary conditions.

In the vacuum regions, the electric field E_y is related to H_z by

$$(\mathrm{curl}\, H)_Y = -\frac{\partial H}{\partial x} = -\frac{i\omega E}{c}. \tag{31}$$

At $x = 0^+$, from (25),

$$\frac{\partial H}{\partial x} = ik(H_1^+ - H_1^-), \tag{32}$$

whence

$$E,(O) = -(ic/\omega)(ik)(H_1^+ - H_1^-) = H_1^+(1 - R), \tag{33}$$

where the reflection amplitude R is defined as

$$R \equiv H_1^-/H_1^+. \tag{34}$$

In the vacuum region 3,

$$E,(d) = -(ic/\omega)(ik)H_3^+ \exp(ikd) = H_1^+ T \exp(ikd), \tag{35}$$

where the transmission amplitude T is defined as

$$T \equiv H_3^+/H_1^+. \tag{36}$$

Now $kd = 2\pi d/\lambda$ by the dispersion relation in vacuum, and by our initial assumption $d \ll \lambda$, so that (35) becomes in this limit

$$E_3(d) = H_1^+ T. \tag{37}$$

In the same limit, the spatial average electric field in the film is

$$\langle E \rangle = [E_1(0) + E_3(d)]/2 = H_1^+(1 - R + T)/2, \tag{38}$$

and the surface current density in the film is

$$J = \sigma \langle E \rangle d = (\sigma d/2)(1 - R + T)H_1^+. \tag{39}$$

Now apply the Ampere law to a thin rectangular path around a unit length of the film

$$(H_1^+ + H_1^-) - H_3^+ \equiv H_1^+(1 + R - T) = 4\pi J/c = (2\pi\sigma d/c)(1 - R + T)H_1^+, \tag{40}$$

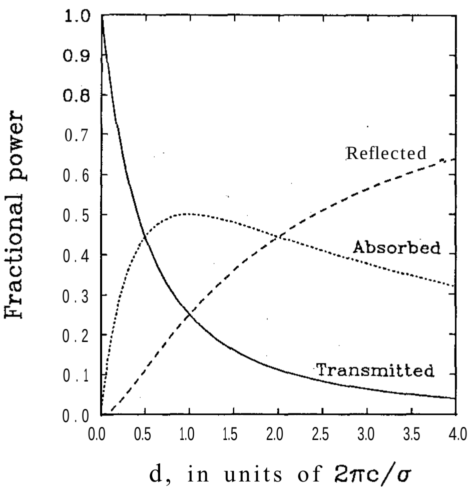

Fig. 2. Transmission, reflection, and absorption of radiation by a thin film, as a function of the film thickness in units of $c/2\pi\sigma$, where σ is the conductivity of the film.

5)

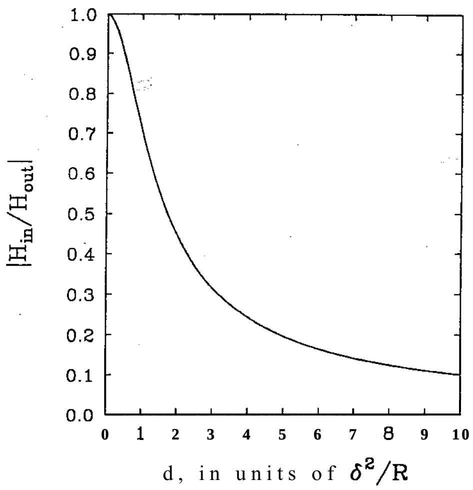

Fig. 1. Screening of the external applied field H_{out} by a cylinder shell of radius $R = 500\delta$, where δ is the skin depth, as a function of the shell thickness d measured in units of δ^2/R.

or

$$|H_i/H_0| = 1/[1 + (Rd/\delta^2)^2]^{1/2}. \tag{9}$$

It follows that the applied field is screened when d $> \delta^2/R$ (see Fig. 1).

We now give the exact solution to this problem, simplifying the method due to L. V. King.[6] (The "physical" solution above has not been found by us in the literature on the subject.) The first Maxwell equation is

$$\nabla\times H = (4\pi/c)\sigma E, \tag{10}$$

apart from the displacement current. The second Maxwell equation is

$$\nabla\times E = (i\omega/c)H, \tag{11}$$

if H has the time dependence exp $(-i\omega t)$. We combine the equations to obtain

$$\nabla^2 H = (2i/\delta^2)H, \tag{12a}$$

where $\delta = (c^2/2\pi\omega\sigma)^{1/2}$ is the skin depth. The analogous equation, arising from the London equations for the superconductor, is

$$\nabla^2 H = (1/\Lambda^2)H, \tag{12b}$$

where Λ is the London penetration depth. For fields with cylindrical symmetry, Eq. (12a) reduces to

$$\frac{d^2H}{dr^2} + \frac{1}{r}\frac{dH}{dr} = -\frac{2i}{\delta^2}H. \tag{13}$$

This applies within the metal, which means for $R<r<R+d$. Let $z = (1-i)r/\delta$, so that (13) becomes the modified Bessel equation of order zero:

$$\frac{d^2H}{dz^2} + \frac{1}{z}\frac{dH}{dz} - H = 0. \tag{14}$$

The general solution of this equation is given by

$$H(z) = a_1 I_0(z) + a_2 K_0(z), \tag{15}$$

where I_0 and K_0 are the modified Bessel functions of order zero.

The tangential components of the magnetic and electric fields are continuous across the metal/vacuum interfaces. The boundary condition at the outer wall is

$$H(R+d-) = H(R+d+) = H_0. \tag{16}$$

At the inner wall we apply the Faraday law to obtain[5]

$$2\pi RE(R^+) = (i\omega/c)(\pi R^2 H_i). \tag{17}$$

Now eliminate E from (10) and (17) to obtain, with curl $H = -dH/dr$, the boundary condition at $r = R^+$ on the inner wall:

$$\frac{dH}{dr} = -i\frac{R}{\delta^2}H. \tag{18}$$

If we write all quantities in terms of the complex variable z, the boundary conditions take the form

$$H(z) = H_o, \ \text{at}\ z = z_o, \qquad \frac{dH}{dz} = \frac{z_i}{2}H, \ \text{at}\ z = z_i, \tag{19}$$

where $z_i = (1-i)R/\delta$ and $z_o = (1-i)(R+d)/\delta$.

It is straightforward to show that the values of a, and a_2 that satisfy the boundary conditions give a solution of the differential equation of the form,

$$H(z) = H_0 \frac{I_0(z)[K_0'(z_i) - \frac{1}{2}z_i K_0(z_i)] - K_0(z)[I_0'(z_i) - \frac{1}{2}z_i I_0(z_i)]}{I_0(z_o)[K_0'(z_i) - \frac{1}{2}z_i K_0(z_i)] - K_0(z_o)[I_0'(z_i) - \frac{1}{2}z_i I_0(z_i)]}. \tag{20}$$

This equation is simplified using standard Bessel function relations,'

$$-(z/2)I_2(z) = I_0'(z) - (z/2)I_0(z),$$
$$-(z/2)K_2(z) = K_0'(z) - (z/2)K_0(z), \tag{21}$$
$$I_0'(z)K_0(z) - K_0'(z)I_0(z) = 1/z.$$

Then the radial dependence of the field in the metal can is given by

$$H(z) = H_0 \frac{I_0(z)K_2(z_i) - K_0(z)I_2(z_i)}{I_0(z_o)K_2(z_i) - K_0(z_o)I_2(z_i)}. \tag{22}$$

The ratio of the inside field H_i to outside field H_o is given by,

$$H_i/H_o = (2/z_i^2)[I_0(z_o)K_2(z_i) - K_0(z_o)I_2(z_i)]^{-1}, \tag{23}$$

an exact result first obtained by King[4] by a more complicated method in which four boundary conditions are used instead of our two. In the limit $R \gg \delta$, we can use the asymptotic forms

$$I_\nu(z) \simeq (1/2\pi z)^{1/2}e^z,$$
$$K_\nu(z) \simeq (\pi/2z)^{1/2}e^{-z}, \tag{24}$$

valid when the $\mathscr{R}$ (z) $\gg 1$. To recover (8) for the screening factor from (23) , we make a linear approximation to I and which may be written as

$$(1+R-T)=\alpha(1-R+T), \tag{41}$$

where a $\equiv 2\pi\sigma d/c$ 'measures d in units of $d_0 \equiv c/2\pi\sigma$.

Finally, we apply the Faraday law to a similar, but perpendicular path. By (33) and (37),

$$E,(O) -E,(d) = H_1^+(1-R) - H_1^+T$$
$$= i\omega d\langle B\rangle/c = i\omega d H_1^+(1+R+T)/2c, \tag{42}$$

whence

$$R+T=1, \tag{43}$$

sinceod $/2c = \pi d/\lambda \ll 1$. We solve (4 1) together with (43) to obtain

$$R=\alpha/(1+a), \quad T=1/(1+a). \tag{44}$$

Both amplitudes R and *Tare* real in this limit.

The reflectance and transmittance are

$$|R|^2=\alpha^2/(1+a)^*, \quad |T|^2=1/(1+a)^*. \tag{45}$$

The power absorbed in the film is proportional to

$$P=1-|R|^2-|T|^2=2\alpha/(1+\alpha)^2, \tag{46}$$

which is a maximum P = 0.5 when a = 1, that is, when $d = c/2\pi\sigma$ (see Fig. 2). When $a \ll 1$ the film is too thin to absorb much of the incident energy, which is chiefly transmitted; when $a \gg 1$ there is an impedance mismatch at the outside of the film and most of the incident energy is reflected. The extreme example of the impedance mismatch effect occurs for films that are thick in comparison with the skin depth: In this limit $d \gg 6$, the reflectance is given by the Hagen–Rubens relation,'

$$|R|^2 = 1-(2\omega/\pi\sigma)^{1/2}. \tag{47}$$

Because there is no transmission in this limit, the power absorption is proportional to $1-|R|^2$ or

$$P=(2\omega/\pi\sigma)^{1/2}=d_0/2\delta. \tag{48}$$

In the infrared, this may be of the order of 0.01, as compared with 0.5 for the maximum power absorbed in the thin film (46). Thus the thick film absorbs much less energy overall than the thin film, all because the thin film offers a good impedance match to free space whereas the thick film is a poor match. Curves of transmission, reflection, and absorption versus d/d_0 are shown in Fig. 2. A poor match means that most of the incident radiation is reflected—although what is not reflected is absorbed and a larger thickness is available for the absorption, the reflection dominates.

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'W. R. Smythe, *Static and Dynamic Electricity* (McGraw-Hill, New York 1943), 3rd ed., Sec. 10.04. Smythe has a superb chapter on eddy currents with a wide collection of problems, some going back to Cambridge Tripo questions. However, because the book is largely devoted to the derivatioı of exact results, the physics may be hidden in complex expressions. Th real physics is, as we show, often much simpler than the exact expression suggest.

[2]J. C. Maxwell, *Electricity and Magnetism* (Oxford U. P., London, 1881) See also Ref. 1, Sec. 10.10.

[3]C. Kittel, S. Fahy, and S. G. Louie, Phys. Rev. B 37, 642 (1988).

[4]M. Tinkham, Phys. Rev. 104,845 (1956). In using (4), because of diffus scattering of electrons from the film surfaces, the conductivity should b taken as if the electron mean free path I- film thickness d when the latter i smaller than the bulk value of Icorresponding to the bulk conductivity σ, The distinction is easily taken into account and we shall assume that it ha been *done* whenever called for.

[5]We assume that H, is constant in space in the vacuum region within tł cylinder-this is equivalent to neglecting the displacement current. We r turn to this point below.

[6]L. V. King, Philos. Mag. 15, 201 (1933). A detailed numerical develoı ment of King's results has been given by S.Shenfeld, IEEE Trans. Electr magn. Compat. **EMC-10,** 29 (1968).

[7]G. N. Watson, *A Treatise on the Theory of Bessel Functions* (Cambridge I P., 1922), pp. 79–80, 202–203.

[8]See C. Kittel, *Introduction to Solid State Physics* (Wiley, New York, 1986 6th ed., p. 315.