NOTES AND DISCUSSIONS

Note on group velocity and energy propagation

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A general proof is given for the equality between group velocity and energy velocity for linear wave propagation in a homogeneous medium with arbitrary spatial and temporal dispersion. © *2000 American Association of Physics Teachers.*

In the series of physics ''Questions'' in the *American Journal of Physics*, K. M. Awati and T. Howes ask for a general proof of the fact that in linear dispersive wave propagation, energy propagates at the group velocity.[1] Several answers to this question have appeared recently,[2] but a general proof has not been provided by any of them. For a stable and nondissipative[3] medium, such a general proof is indeed available from classical, continuum electrodynamics of media.[4] It can be readily argued that this type of proof can be carried out for any other (than electrodynamic) description of a stable, nondissipative, linear (or, more generally, linearized dynamics of a) medium. In an electrodynamic formulation, one chooses to eliminate the ''mechanical'' field variables in favor of the electromagnetic fields; one can easily visualize doing the opposite. For exposing the simplest electrodynamic proof, I will focus on a nonmagnetic medium ($\vec{B} = \mu_0 \vec{H}$) which is arbitrarily (spatially and temporally) dispersive—a plasma. The generalization to linear waves in magnetic media is straightforward.

The continuum electrodynamics of a plasma-like medium is described by Maxwell's equations for the electromagnetic fields $\vec{E}$ and $\vec{H}$, wherein the collective ''mechanical'' dynamics of the medium as a function of $\vec{E}$ and $\vec{H}$ are expressed by electric current and electric charge densities $(\vec{J}, \rho)$. The latter, expressed as functions of $\vec{E}$ and $\vec{H}$, are what one can call the electrodynamic response functions of the medium. For a plasma-like medium, since $\vec{B} = \mu_0 \vec{H}$, Faraday's equation provides a way of eliminating $\vec{H}$ in favor of $\vec{E}$, so that the response functions are only functions of $\vec{E}$. In addition, since $\vec{J}$ and ρ are related by the continuity equation, the mechanical dynamics, regardless of the particular model chosen for describing the dynamics, can be expressed by a single ''electrical'' response (or ''influence'') function, e.g., the conductivity tensor $\vec{\sigma}$, or the susceptibility tensor $\vec{\chi}$, or the permittivity tensor $\vec{K}$, or the dielectric tensor $\vec{\epsilon}$—all of these being related to each other.

In a homogeneous medium of the type we are considering, the most general linear response function is one which expresses both spatial and temporal dispersion through a convolution integral in both space and time. Thus, for example, the current density $\vec{J}(\vec{r}, t)$, at a point location $\vec{r}$ and time t, depends upon the time-history and location-neighborhood (consistent with causality and relativity) of the electric field $\vec{E}(\vec{r}, t)$ through the space–time conductivity tensor influence function. The Fourier–Laplace transform of this convolution relationship is expressed by the conductivity tensor function of wave vector $\vec{k}$ and frequency ω as

$$\vec{J}(\vec{k}, \omega) = \vec{\sigma}(\vec{k}, \omega) \cdot \vec{E}(\vec{k}, \omega). \tag{1}$$

The other linear response functions have similar interpretations, and are simply related to each other: $\vec{\chi}(\vec{k}, \omega) = \vec{\sigma}(\vec{k}, \omega)/(-i\omega\epsilon_0)$; $\vec{K}(\vec{k}, \omega) = \vec{I} + \vec{\chi}(\vec{k}, \omega)$; $\vec{\epsilon}(\vec{k}, \omega) = \epsilon_0 \vec{K}(\vec{k}, \omega)$. The Fourier–Laplace transform of Maxwell's equations for the self-consistent electromagnetic fields are then

$$\vec{k} \times \vec{E} = \omega \mu_0 \vec{H} \tag{2}$$

and

$$\vec{k} \times \vec{H} = -\omega \epsilon_0 \vec{K}(\vec{k}, \omega) \cdot \vec{E}. \tag{3}$$

Taking $\vec{k} \times$ (2) and using (3) to eliminate $\vec{H}$, one finds the homogeneous set of equations for $\vec{E}$:

$$\vec{D}(\vec{k}, \omega) \cdot \vec{E} = 0, \tag{4}$$

where the dispersion tensor $\vec{D}$ is

$$\vec{D}(\vec{k}, \omega) = \vec{k}\vec{k} - k^2 \vec{I} + \frac{\omega^2}{c^2} \vec{K}(\vec{k}, \omega). \tag{5}$$

For nontrivial solutions of (4),

$$\det[\vec{D}(\vec{k}, \omega)] \equiv D(\vec{k}, \omega) = 0, \tag{6}$$

which is the dispersion relation giving, e.g., $\omega(\vec{k})$. These are the natural modes of the system, with fields whose space–time dependence $\exp[i(\vec{k} \cdot \vec{r} - \omega t)]$ is constrained by the dispersion relation (6). Natural modes that are purely propagating waves are those for which solutions of (6) entail real $\vec{k} = \vec{k}_r$ and real $\omega = \omega_r$, i.e., $\omega_r(\vec{k}_r)$. The group velocity for such waves is given by

$$\vec{v}_g = \frac{\partial \omega_r}{\partial \vec{k}_r}. \tag{7}$$

In a dissipation-free medium, the permittivity tensor is Hermitian for real $\vec{k}$ and real ω, $\vec{K}(\vec{k}_r, \omega_r) = \vec{K}_h$. In a linearly stable and dissipation-free medium, the direction of signal propagation is given by the direction of $\vec{v}_g$.[5]

In order to determine the velocity with which energy is transported, one needs to first determine the appropriate formulation of energy and energy flow in a space–time dispersive medium. For the purely propagating wave modes in a plasma-like medium, one can show that the average (in space or time) energy density is given by[6]

$$\langle w \rangle = \frac{\mu_0}{4} |\vec{H}|^2 + \frac{\epsilon_0}{4} \vec{E}* \cdot \frac{\partial(\omega_r \vec{K}_h)}{\partial \omega_r} \cdot \vec{E}, \quad (8)$$

where the first term is clearly the average magnetic energy density and the second term is the average energy density in the electric field and in all of the collective ''mechanical'' fields. One can also show that the average energy flow density is given by[6]

$$\langle \vec{s} \rangle = \text{Re}\left(\frac{1}{2} \vec{E} \times \vec{H}* \right) - \frac{\epsilon_0}{4} \omega_r \vec{E}* \cdot \frac{\partial \vec{K}_h}{\partial \vec{k}_r} \cdot \vec{E}, \quad (9)$$

where the first term is the average electromagnetic (Poynting) energy flow density and the second term is the average collective ''mechanical'' energy flow density. Using (8) and (9), one can define an energy flow velocity for a natural wave $\omega_r(\vec{k}_r)$:

$$\vec{v}_e = \frac{\vec{s}_k}{w_k}, \quad (10)$$

where $\vec{s}_k = \langle \vec{s} \rangle|_{\omega_r(\vec{k}_r)}$ and $w_k = \langle w \rangle|_{\omega_r(\vec{k}_r)}$ are the average wave energy flow density and wave energy density, respectively.

The proof that (7) and (10) are equal to each other proceeds as follows. Consider Maxwell's equations (2) and (3), for $\vec{k} = \vec{k}_r$, $\omega = \omega_r$, and $\vec{K}(\vec{k}, \omega) = \vec{K}_h(\vec{k}_r, \omega_r)$,

$$\vec{k}_r \times \vec{E} = \omega_r \mu_0 \vec{H}, \quad (11)$$

$$\vec{k}_r \times \vec{H} = -\omega_r \epsilon_0 \vec{K}_h(\vec{k}_r, \omega_r) \cdot \vec{E}. \quad (12)$$

As followed from (2) and (3), these entail the dispersion relation

$$\det\left[\vec{k}_r \vec{k}_r - k_r^2 \vec{I} + \frac{\omega_r^2}{c^2} \vec{K}_h(\vec{k}_r, \omega_r) \right] \equiv D_h(\vec{k}_r, \omega_r) = 0, \quad (13)$$

giving $\omega_r(\vec{k}_r)$, and thence the group velocity,

$$\vec{v}_g = \frac{\partial \omega_r}{\partial k_r} = \frac{(-\partial D_h/\partial \vec{k}_r)_{\omega_r(\vec{k}_r)}}{(\partial D_h/\partial \omega_r)_{\omega_r(\vec{k}_r)}}. \quad (14)$$

In addition, consider the variation of (11) and (12) with respect to $\vec{k}_r$ and ω_r:

$$(\delta \vec{k}_r) \times \vec{E} + \vec{k}_r \times (\delta \vec{E}) = (\delta \omega_r) \mu_0 \vec{H} + \omega_r \mu_0 (\delta \vec{H}), \quad (15)$$

$$(\delta \vec{k}_r) \times \vec{H} + \vec{k}_r \times (\delta \vec{H}) = -\epsilon_0 \delta(\omega_r \vec{K}_h) \cdot \vec{E} - \omega_r \epsilon_0 \vec{K}_h$$
$$\cdot (\delta \vec{E}). \quad (16)$$

Dot-multiplying (15) by $\vec{H}*$, (16) by $-\vec{E}*$, the complex conjugate of (11) by $-(\delta \vec{H})$, the complex conjugate of (12) by $(\delta \vec{E})$, and adding these equations, one obtains

$$(\delta \vec{k}_r) \cdot \langle \vec{s} \rangle = (\delta \omega_r) \langle w \rangle \quad (17)$$

from which one immediately finds

$$\frac{\partial \omega_r}{\partial \vec{k}_r} = \frac{\vec{s}_k}{w_k}. \quad (18)$$

This proves the equality of the group velocity and energy velocity, $\vec{v}_g = \vec{v}_e$, for purely propagating waves in a linear, generally dispersive, and nondissipative ''electric'' medium, like a plasma. Note that, in the above proof, no specific model of the linear, loss-free, dispersive dynamics had to be specified; the result (18) is thus valid for any linear, dispersive dynamics of a loss-free medium.

Two remarks are in order. First, in relation to the assumption of a nondissipative medium, the Kramers–Krönig relations for a dispersive medium require that the permittivity tensor, $\vec{K}(\vec{k}_r, \omega_r)$, have both a Hermitian and an anti-Hermitian part. The relative magnitudes of these parts can, however, vary from region to region in $(\vec{k}_r, \omega_r)$ space. Weakly damped waves $[|\omega_i(k_r)| \ll |\omega_r(k_r)|]$, for which (18) holds, exist in regions of $(\vec{k}_r, \omega_r)$ where the anti-Hermitian part of $\vec{K}$ is small compared to its Hermitian part so that $\omega_r(\vec{k}_r)$ is essentially determined by (13). Second, group velocity (as its name is intended to remind us) applies to the velocity of a group of waves—a wave packet—and by (18) this must also be true for energy velocity in a loss-free, dispersive medium.[7] Allowing for the wave fields $[\exp(i\vec{k}_r \cdot \vec{r} - i\omega_r t)]$ to have amplitudes that vary slowly in space and time (slowly compared to the fast scales of, respectively, $\vec{k}_r$ and ω_r), their velocity is also found to be given by (14). In addition, averaged (on the fast scales of either $\vec{k}_r$ or ω_r) energy and energy flow densities are again found to be given by (8) and (9), respectively, and one can show that (17) and (18) also hold for such wave fields with slow space–time amplitude modulations. A detailed proof of the above, including the account of weak dissipation, is given by Bers (Ref. 8).

Several concluding remarks are also in order. The above derivation carries through for a weakly inhomogeneous and/or weakly time-varying medium as long as geometrical optics is applicable to describe the wave propagation.[8] The proof of (18) can also be carried out for a weakly dissipative or weakly unstable medium.[8] However, in a linearly unstable medium [i.e., in which for some $\vec{k}_r$, $\omega(\vec{k}_r) = \omega_r(\vec{k}_r) + i\omega_i(\vec{k}_r)$ has $\omega_i(\vec{k}_r) > 0$], the group velocity direction for purely propagating wave modes having $\omega(\vec{k}_r) = \omega_r(\vec{k}_r)$, for some other $\vec{k}_r$, may not be the same as the direction of signal propagation.[9]

[1]K. M. Awati and T. Howes, Am. J. Phys. **64** (11), 1353 (1996).
[2]K. T. McDonald, C. S. Helrich, R. J. Mathar, S. Wong, and D. Styer, Am. J. Phys. **66** (8), 656–661 (1998).
[3]Well-defined wave propagation in a dispersive medium is understood to take place in a regime of weak dissipation where the modes are characterized by frequencies and wave numbers that are essentially real. It is only in these regimes that the concepts of group velocity, time-, or space-averaged energies and their flows, and thus energy velocity, are well-defined. In thus considering these concepts, one can idealize the situation by considering the medium to be ''nondissipative'' as we do in the following.
[4]Covariant electrodynamic formulations of average wave energy and momentum and their flows in linear media were given for temporally dispersive media by S. M. Rytov, JETP **17**, 930 (1947), and generalized to spatially and temporally dispersive media by M. E. Gertsenshtein, *ibid.* **26**, 680 (1954). Independently, a simpler formulation of average wave energy and energy flow in linear media with spatial and temporal dispersion was

given by A. Bers in 1962; see W. P. Allis, S. J. Buchsbaum, and A. Bers, *Waves in Anisotropic Plasmas* (MIT, Cambridge, MA, 1963), Sec. 8.5. As shown in this last reference, the equality between group velocity and energy velocity follows simply from a variational form of Maxwell's equations for an arbitrarily dispersive medium; this general proof is the basis of this note.

[5]See, for example, L. Brillouin, *Wave Propagation and Group Velocity* (Academic, New York, 1960), Chap. I.

[6]For a loss-free, isotropic, and only temporally dispersive medium [$\vec{\vec{K}}_h = \vec{I} K(\omega_r)$ and real], (8) was first identified by L. Brillouin in 1932; see Ref. 5 above, Chap. IV. For a loss-free, anisotropic medium with spatial and temporal dispersion [$\vec{\vec{K}}_h = \vec{\vec{K}}_h(\vec{k}_r, \omega_r)$], (8) and (9) were derived independently by several authors; see Ref. 4 above, and the treatment by Bers in Ref. 8 below.

[7]Note that for propagation in regions of strong absorption, it is well-known that this equality breaks down; see Chap. V in Ref. 5 above. In such regions $\omega(\vec{k}_r) = \omega_r(\vec{k}_r) + i\omega_i(\vec{k}_r)$, and $|\omega_i(\vec{k}_r)|$ is not small compared to $|\omega_r(\vec{k}_r)|$. Then, calculating $\vec{v}_g$ from $\omega_r(\vec{k}_r)$ can give values for v_g that exceed the speed of light and hence, in such regions, the group velocity no longer represents the velocity of energy flow; energy cannot travel at such speeds.

[8]A. Bers, ''Linear Waves and Instabilities,'' in *Plasma Physics—Les Houches 1972*, edited by C. DeWitt and J. Peyraud (Gordon and Breach, New York, London, Paris, 1975), Secs. II and VII; I. B. Bernstein, ''Geometric Optics in Space- and Time-Varying Plasmas,'' Phys. Fluids **18**, 320 (1975).

[9]A. Bers, ''Space-Time Evolution of Plasma Instabilities—Absolute and Convective'' in *Handbook of Plasma Physics*, general editors, M. N. Rosenbluth and R. Z. Sagdeev, *Vol. I Basic Plasma Physics*, volume editors, A. A. Galeev and R. N. Sudan (North-Holland, Amsterdam, 1983), Chap. 3.2, Sec. 3.2.3.

Analysis of doubly excited symmetric ladder networks

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A simple procedure to determine the effective resistance between the center and a vertex of an n-sided polygon made of resistors is presented. © *2000 American Association of Physics Teachers.*

In a recent paper[1] Sidhu presented a procedure to determine the effective resistance between the center and a vertex of an n-sided polygon made of resistors. The resistance connected between the center and a vertex of the polygon is 1 Ω, whereas the resistance connected between any two successive vertices is $2G$ Ω. In this note, we present a simple alternative procedure that is easier to comprehend. The units for current (ampere) and voltage (volt) are not specified with respect to each variable in the text below.

Consider the ladder network[2] shown in Fig. 1. We let $E_1 = E_2 = E$. From considerations of symmetry, $I_0 = 0$. Further, $I_1 = I_2$, $I_3 = I_4$, $I_5 = I_6$, $I_7 = I_8$, $I_9 = I_{10}$, and $I_{11} = I_{12}$. We propose to determine E for which $I_1 = 1$. Under such a condition, $V_{CN} = 1$ and $I_3 = I_0 + I_1 = 1$.

Further,

$$V_{BN} = V_{BC} + V_{CN} = (2G + 1),$$

$$I_5 = (2G + 1),$$

$$I_7 = I_3 + I_5 = (2G + 2),$$

$$V_{AN} = E = I_7(2G) + V_{BN} = (4G^2 + 6G + 1),$$

$$I_9 = V_{AN}/2 = (2G^2 + 3G + 0.5),$$

$$I_{11} = I_7 + I_9 = (2G^2 + 5G + 2.5).$$

If we use the *same source* to provide the two voltage inputs of the network as shown in Fig. 2, the current delivered by that source would be $I_{11} + I_{12}$. Thus $I_s = (4G^2 + 10G + 5)$. It may be seen that the network shown in Fig. 2 is a five-sided polygon of resistances defined in Ref. 1. Hence, $R_5 = E/I_s = V_{AN}/I_s = (4G^2 + 6G + 1)/(4G^2 + 10G + 5)\,\Omega$.

The resistance R_{2N+1} for any N, may be computed by considering appropriate ladder network. It should have ($2N + 1$) resistances of value $2G$ Ω, $2N$ resistances of value 1 Ω, and two 2 Ω resistances (one at each end). Note that the computation of successive voltages and currents is a recursive procedure. We now consider the determination of R_{2N}. The network used for the determination of R_{2N} is derived from the network used for the determination of R_{2N+1}. The ''mid-section'' is modified as follows: Each of the two resistances from the vertex to the center is changed to 2 Ω. The resistance between the vertices is set to zero. The network for R_4 is obtained from the circuit shown in Fig. 1. The element values are as shown in Fig. 3.

We analyze the circuit in Fig. 3 as earlier. We note that I_0 is zero. We propose to determine E for which $I_1 = 1$. Under such a condition, $V_{CN} = 2$. Then $I_3 = I_0 + I_1 = 1$.

Further,

$$V_{BN} = V_{BC} + V_{CN} = (2G + 2),$$

$$I_5 = (2G + 2),$$
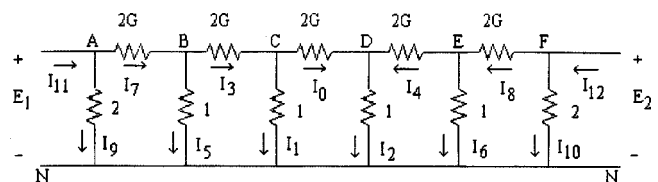
$$I_7 = I_3 + I_5 = (2G + 3),$$
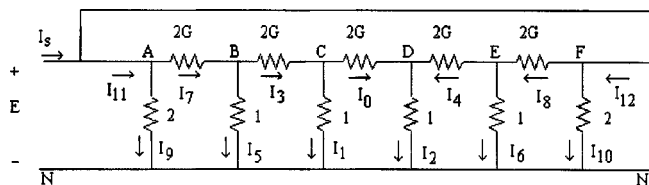

Fig. 1. Ladder network.

Fig. 2. Doubly excited symmetrical ladder network.

Fig. 3. Modification of Fig. 1.

$$V_{AN} = E = V_{BN} + I_7(2G) = (4G^2 + 8G + 2),$$

$$I_9 = V_{AN}/2 = (2G^2 + 4G + 1),$$

$$I_{11} = I_7 + I_9 = (2G^2 + 6G + 4).$$

If we use the *same source* to provide the two voltage inputs of the network as shown in Fig. 2, the current delivered by that source would be $I_{11} + I_{12}$. Thus $I_s = (4G^2 + 12G + 8)$. Thus,
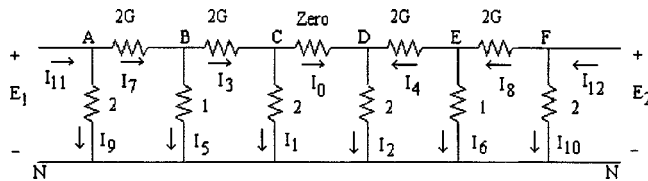
$$R_4 = V_{AN}/I_s = (4G^2 + 8G + 2)/(4G^2 + 12G + 8)\,\Omega$$

$$= (2G^2 + 4G + 1)/(2G^2 + 6G + 4)\,\Omega.$$

The procedure outlined in the note is suitable for classroom use. For a given value of G (say $G = 1$), the effective resistance R_9 or R_8 can be computed in a short time.

[1]Satinder S. Sidhu, ''Polygons of unequal resistors,'' Am. J. Phys. **62**, 815–816 (1994).
[2]M. E. Van Valkenberg, *Network Analysis* (Prentice–Hall, Englewood Cliffs, NJ, 1964).

THE METRIC SYSTEM

The only case for the French metric system is that it has become sufficiently universal so that there are real advantages in making it completely universal. It cannot claim the slightest scientific validity as its units are not based on any natural units and are psychologically not even particularly convenient. The decimal system is one of the less convenient systems of counting, though by no means the worst. The only argument for it is that when it doesn't really matter what we do, it is convenient to have everybody do the same thing, so let's all join the party.

Kenneth Boulding, ''Numbers and Measurement on a Human Scale,'' in *The Metric Debate*, edited by David F. Bartlett (Colorado Associated University Press, Boulder, 1980), p. 64.