Student experiments on parametric resonance

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Some experiments on parametric effects, to be constructed by the students themselves, are described together with the discussion and questions given. Parametric oscillations were produced on a string, in a resonance circuit and a spring pendulum. The importance of phase relations for excitation of the oscillations and of nonlinearities in their stabilization was emphasized. The experiments were so organized that the students would meet unknown phenomena and problems; actually we discovered an unexpected recurrence effect in the spring pendulum.

I. INTRODUCTION

The experiments discussed in this paper are intended to be designed partly by the students themselves. The primary aim is to show how complicated phenomena such as parametric interactions and nonlinearities can be introduced to the students and discussed in a very simple manner.

The opportunity to prepare the experiments occurred when departments at Chalmers University of Technology were asked to perform introductory laboratory experiments with the new students. The students' knowledge of mathematics is limited to basic calculus, but if one wants to stimulate their interest the experiments must concern modern and active physics. It seems worthwhile to look for suitable experiments since many laboratory courses are now modernized along these lines.

Laboratory courses in physics traditionally repeat classical experiments and measurements. However, many of these experiments, such as Cavendish's weighing the earth or Millikan's oil drop experiment are essentially demonstrations and better suited to the lecture room, since the student's work does not add to his understanding of the phenomenon. Students who hope to find in physics something new are not satisfied by measuring well-known constants and it is often tempting to omit measurements which fail to produce the expected result. To avoid this possibility the experiments were based on physical principles the students would not know, but which are amenable to simple mathematical explanations. Apart from parametric resonance, experiments were also made on different aspects of Huygen's principle. The idea was to let different groups demonstrate the principles in quite different physical contexts; finally all groups demonstrated their experiments to the others. The element of exploration, which is missing in most laboratory experiments, was present, since the effects were often quite hard to find; the students also had to discover the form and parameters of the experiment which produced the best demonstration of the effect.

It is interesting that most students prefer to take part in building up the experiment instead of coming to a ready apparatus; it only takes good nerves on the part of the teacher since not all effects are easily reproducible. Questions and problems were given to the students during the course to fix the important ideas at each stage and are discussed in the text.

The three experiments given on parametric resonance (the spring pendulum, parametric excitation on a string, and in an electric circuit) together with the introduction given to the students will be described.

II. TEACHER'S INTRODUCTION

The idea of parametric amplification was introduced by demonstrating how pendulum oscillations amplify when the pendulum length l is suitably varied. Children's swings[1] and the more exotic incense swing[2] are interesting applications to mention. Since students believe firmly in conservation of energy, they are immediately convinced of the possibility of introducing energy by pulling the rope when it feels heavy and slackening when it feels light. The net energy gain is easily calculated for small upward jerks Δl at the middle position and similar drops at maximum amplitude.[1] The gain is $\Delta E = 6E\Delta l/l$ per period which gives a nearly exponential growth. It is important to stress the *phase relation* involved in the energy gain: one should do work on the system, i.e., vary the parameter l, when it is hardest during the oscillation.

Maximum work can be done on the system at two symmetrical points during the oscillation, so if it is necessary to vary the parameter periodically it should be done at the frequency $2\omega_0$, twice the oscillation frequency ω_0. It may be confusing to the students and must not be stressed unduly, but depending on the form of the oscillations and the parameter variations there are many frequency ratios that give over all energy gain; when both are sinusoidal the well-known condition $\omega = 2\omega_0/n$ occurs. The natural oscillation will then adjust its phase so that a positive energy transfer takes place.

III. SPRING PENDULUM

A particularly impressive demonstration of the effect is given by the spring pendulum.[3] The weight on a spring is adjusted so that the spring frequency ω_s is twice the pendulum frequency ω_p. If the spring oscillation is initiated the pendulum length varies with frequency $\omega_s = 2\omega_p$ and the pendulum oscillation quickly grows parametrically according to the discussion in Sec. II. From this mode energy flows back to the spring oscillations, which are excited *subharmonically* by the pendulum movement, $\omega_p = \omega_s/2$. For a well-adjusted spring this process will repeat itself several times which makes a spectacular impression.

The spring is best adjusted to the condition $\omega_s = 2\omega_p$ by comparing the oscillation times for small linear oscillations;

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since only *relative changes* in ω are interesting, this is a good point to derive the oscillation frequencies by dimensional analysis. Part of the mass is in the spring and $\omega_s \sim m^{-1/2}$ and $\omega_p \sim l^{-1/2}$ are only used for rough guesses. The actual coupling between the two oscillations is provided by the small nonlinear terms in the equations[3]

$$\ddot{x} + \omega_p^2 x = \lambda x z, \quad (1)$$

$$\ddot{z} + \omega_s^2 z = (\lambda/2) x^2. \quad (2)$$

It was effective to have the students compare this system with the double pendulum and the pendulum with free support,[4] which are also conservative oscillating systems with two degrees of freedom. They are however *linear* and though the amplitude may grow rapidly when the oscillations are moving into phase, the solution is a superposition of sines and cannot show the nearly exponential parametric growth.

This example was given to the students to demonstrate the *usefulness* of nonlinearities, which are usually treated as unpleasant complications among the linear solutions; in the following sections their importance in stabilizing the parametric growth will be emphasized.

Finally, it must be mentioned that of course not all aspects of this system can be understood from a qualitative picture. For instance, we noticed that energy seems to pass from the spring mode to the pendulum mode and back again in a far more complete manner than one would expect from the simple parametric picture; this point required a closer investigation of Eqs. (1) and (2).[5]

IV. PARAMETRIC WAVES ON A STRING

This form of parametric oscillation was mentioned by Rayleigh in his *Theory of Sound.*[6] He referred to the experiment by Melde who excited *transversal* waves on a string by stretching it *longitudinally* with a tuning fork. (Rayleigh also gave many other examples and a delightful reference to Faraday, who experimentally discovered that his "crispations," capillary waves, had frequencies ½ of the exciting frequency, i.e., were parametric.)

We constructed the experiment somewhat differently in order to demonstrate clearly the details of the oscillations, particularly the phase relation (Fig. 1). The string was a rubber band which made large amplitudes possible at low frequencies (about 10 Hz). The relative change in the tension S (the parameter of the oscillation) must be sufficiently large to overcome losses; this was achieved very simply by mounting a screw excentrically on a wheel and threading a thin wire over it and through the string. The speed of the wheel (which is actually used to chop laser light) could be regulated rather precisely electronically; this is an advantage for a weak motor, since the parametrically excited wave will affect the speed of the wheel.

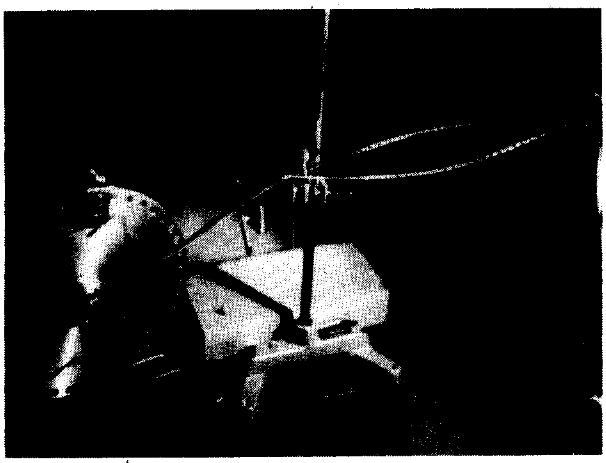

Fig. 1. Parametric oscillations on a string photographed in stroboscopic light: *two* strings are seen oscillating when the driving wheel is frozen. The fine wire between the string and the excentrically mounted screw is invisible.

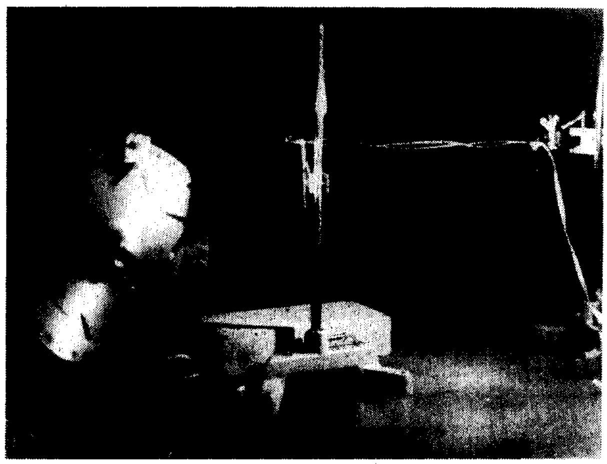

Fig. 2. Parametrically produced oscillations in a higher mode.

In stroboscopic light the stabilized wave form and its phase relation to the pulling force are demonstrated with convincing clarity (Figs. 1 and 2); the slow oscillations are spectacular and much of the effect, as so often, depends on the "artistic" side of the arrangement.

Oscillations are only excited if an integer number of half-wavelengths covers the length of the string. One can try to satisfy this condition by plucking the string (measuring the resonance frequency with the stroboscope) and adjusting the length to a good resonance. The students were then worried by the fact that the variation in tension ΔS given by the wheel was by no means small: at the two extreme positions the string was very taut and almost slack, respectively. How can the string "remember" its resonance frequency $\omega_0 = (S/\rho)^{1/2}$ under such circumstances? This question will be treated quantitatively in Sec. V, but the answer is essentially that the string moves to a new frequency which depends also on ΔS. Consequently a better way to find the resonance is to run the wheel at constant speed, varying tension and length until the parametric oscillation occurs. In this way a motor with constant speed can be used.

Under stroboscopic light the phase relationship between the wheel and string is clearly followed. Other waves often appear at the exciting frequency so one must check that *two* strings are seen when the wheel is frozen (Fig. 1). The phase relation between the wave and the wheel is not quite so easy to explain as one would expect; the string should be pulled when the wave has maximum amplitude and slackened at minimum to do work on the system. However, when the tension is very unsymmetrical the taut position will alone transfer energy and determine the energy balance. The phase then seems to be unstable and makes sudden jumps, while the wave has a constant amplitude.

The amplitude of oscillation was quite large and it was interesting to discuss with the students possible saturation mechanisms in a qualitative way. Most students favor different loss mechanisms, such as friction against the air and internal heating. Judging from the amplitude, however, it is much easier to argue for nonlinear effects: for large amplitudes the tension is a nonlinearly increasing function of string length and nonlinear terms which are usually neglected in the string equation must be included (cf. the large angle between the string and the equilibrium position in Fig. 1). The nonlinear terms not only make it impossible to pass a certain amplitude, they also destroy the excitation mechanism by introducing frequency shifts in the wave. In the electric example we could study these saturation effects in a quantitative way.

V. PARAMETRIC CIRCUIT OSCILLATIONS

This experiment is very simple in principle: a variable capacitance diode is used to vary the capacitance in a resonance circuit at twice the resonance frequency ω_0. In practice it turned out that with the material we had available the parametric effect was just about possible to produce. This was an advantage because it made it necessary for the students to do several simple and instructive calculations to optimize the circuit.

We used a BB104 silicon variable capacitance diode with a maximum capacitance 40 pF (Fig. 3). A sufficient circuit Q value was possible only at maximum oscillator frequency so $f_0 \approx 500$ kHz. The circuit was designed by the students, who usually had had some experience with electronic elements, to provide both a driving voltage at $2f_0$ and the reverse voltage for the diode. After some trial and error we arrived at the final circuit shown in Fig. 4. Actually, the circuit contained many stray capacitances at megahertz frequencies so it became necessary to use both capacitances in the diode to reach a satisfactory ratio $\Delta C/C$. With damping in the circuit this ratio must be big enough to overcome losses. The equation for the circuit current is

$$\ddot{I} + (R/L)\dot{I} + (LC)^{-1}I = 0. \quad (3)$$

The condition for parametric growth is[7]

$$\Delta\omega_0/\omega_0 > 2R/L\omega_0 = 2/Q, \quad (4)$$

where $\Delta\omega_0 \sin 2\omega_0 t$ is the variation of ω_0 due to the modulation in C; $\omega_0^2 = 1/LC$ and consequently $\Delta\omega_0/\omega_0 = (1/2)\Delta C/C$ so

$$\Delta C/C > 4/Q. \quad (5)$$

These estimates indicated that a low resistance 1-mH ferrite coil would satisfy the condition with $C \approx 100$ pF calculated from the resonance frequency. The parametric oscillations were found as predicted and Fig. 5(a) shows the superposition of the oscillator signal and the excited oscillations with period twice as large.

Several statements about nonlinear saturation in Secs. I–IV were checked on the oscilloscope. The oscillations were clearly limited by the nonlinearity because the voltage

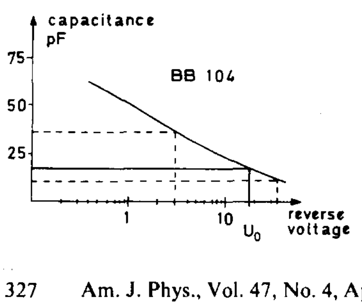

Fig. 3. Characteristic of the capacitance diode BB 104.

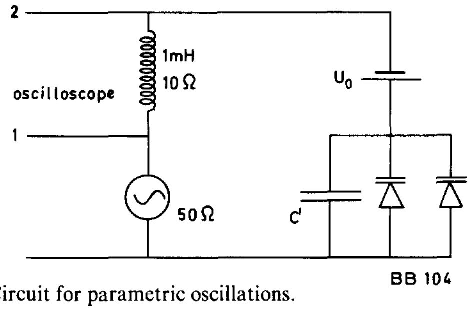

Fig. 4. Circuit for parametric oscillations.

amplitude was always so big that it took C to the steep part of the characteristic. A very nice way of demonstrating this effect is to ask the students how one should change the reverse voltage U_0 in order to increase the oscillation amplitude for a given oscillator signal. The answer usually was that one should improve ΔC by reducing the reverse voltage (cf. Fig. 3); but such an answer, quite excusably of course, confuses the *excitation* of the oscillation with the *saturation mechanism*; once the parametric growth has started it would go on to infinity in the linear approximations for any sufficient ratio $\Delta C/C$. Instead the oscillation amplitude will increase when the reverse voltage is increased, because we are then on a part of the characteristic where C is a more linear function of U (note that U has logarithmic scale in Fig. 3); consequently deviations may become greater before the nonlinearities set in.

The independence of excitation efficiency and maximum amplitude also appears when the parametric oscillation is started with the least possible oscillator signal. Once the parametric oscillation has started the oscillator amplitude may be decreased well below this level without the wave being much affected.

It is also very interesting that the parametric oscillations continue and even increase in amplitude if the oscillator frequency is decreased, but quickly disappear when the frequency is increased. This is due to the existence of a nonlinear frequency shift and may be considered the decisive proof that the stabilization is caused by nonlinearities. The frequency shift $\delta\omega$ depends on oscillation amplitude u as $\delta\omega \sim u^2$ and it was quickly confirmed that the possible frequency range increased with amplitude. The effect was so interesting that a quantitative analysis was made.

A simple equation including nonlinear terms is obtained by considering the oscillating charge q of the capacitance;

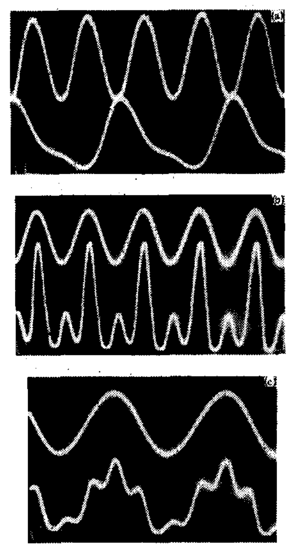

Fig. 5. (a) Oscillator signal (upper trace) and the capacitance voltage with a parametrically excited component at half the oscillator frequency (lower trace); 10 V/unit scale. (b) Decreasing the oscillator frequency (upper trace) increases the amplitude of the parametric oscillations (lower trace). (c) Subharmonically generated oscillations.

the stray capacitances may be included in the constant capacitance C' used to vary ω_0. The total capacitance of C' and the diode is a nonlinear function of the voltage U (Fig. 3), which we expand around the working point U_0 ($u = U - U_0$):

$$C = C_0 + au + bu^2 + \cdots. \quad (6)$$

Now $q = Cu$ but the reverse relation is needed

$$u = q(1/C_0 + Aq + Bq^2 + \cdots). \quad (7)$$

A simple inversion of the series shows that

$$A = -a/C_0^3, \qquad B = -b/C_0^4 + 2a^2/C_0^5. \quad (8)$$

Now

$$u + L\ddot{q} + R\dot{q} = V\cos(2\omega_0 + \epsilon)t,$$

i.e.,

$$\ddot{q} + (\omega_0/Q)\dot{q} + \omega_0^2 q + \alpha q^2 + \beta q^3 = f\cos(2\omega_0 + \epsilon)t, \quad (9)$$

where

$$\omega_0^2 LC_0 = 1, \qquad \alpha = A/L, \qquad \beta = B/L, \qquad f = V/L.$$

The nonlinear terms give a frequency shift[7]

$$\delta\omega = [3\beta/8\omega_0 - (5\alpha^2/12\omega_0^3)]q_0^2 \equiv \kappa q_0^2, \quad (10)$$

where q_0 is the amplitude of oscillations. The stabilized amplitude becomes ($\kappa < 0$; cf. Sec. 29 of Ref. 7)

$$q_0^2 = (1/\kappa)\{(1/2)\epsilon - [(\alpha f/6\omega_0^3)^2 - (\omega_0/Q)^2]^{1/2}\}. \quad (11)$$

To use these results numerically we notice that the characteristic in Fig. 3 is well described by

$$C = -C_1 \ln U/U_1. \quad (12)$$

Expanding around U_0,

$$C = C_1[\ln U_1/U_0 - u/U_0 + (1/2)(u/U_0)^2 + \cdots], \quad (13)$$

we identify the coefficients in (6). Inserting into (10),

$$\kappa = (\omega_0 C_1/9U_0^2 C_0^3)(C_1/C_0 - 27/16). \quad (13)$$

We measured $U_1 = 108$ V, $C_1 = 9.7$ pF for $U_0 = 18$ V in Fig. 3; $C_0 \approx 100$ pF so the frequency shift is always negative in agreement with observation. The amplitude becomes explicitly

$$q_0^2 = \frac{1}{\kappa}\left\{\frac{1}{2}\epsilon - \omega_0\left[\left(\frac{C_1 V}{6C_0 U_0}\right)^2 - \frac{1}{4Q^2}\right]^{1/2}\right\}. \quad (14)$$

Inserting $V \approx 15$ V and $Q = 52$ we confirmed that the root is real which is the condition for parametric oscillations. The amplitude of the oscillation is

$$u = q/C_0 = 7.4U_0[0.010 - \epsilon/2\omega_0]^{1/2}, \quad (15)$$

after introducing the numerical values. Putting $\epsilon = 0$ we obtain $u \approx 13$ V which should be compared with the amplitude in Fig. 5(a) $u \approx 10$. {Superimposed on the parametric oscillation in Fig. 5(a) is the oscillator signal over the diode, $V/3 \cos[(2\omega_0 + \epsilon)t + \pi]$; the voltage difference at two minima of the oscillator signal immediately gives the peak-to-peak voltage 20 V of the parametric oscillation.} This is very good agreement, probably slightly fortuitous in view of the rough approximations; e.g., we have not taken into account the strong asymmetry in the characteristic, Fig. 3, the same problem which occurred when we excited the string. Anyway, the theory described the parametric behavior very well. According to (15) the amplitude will increase further when the frequency is decreased. This was indeed confirmed [Fig. 5(b)] and it was possible to give the diode forward voltage for part of the period. It is, however, typical of parametric phenomena that this must be done by decreasing the frequency continuously from the frequency where the instability occurred. It is impossible to reach these high amplitudes discontinuously. We could not find a continuous decrease of the stabilized amplitude to zero by increasing the frequency, however, as pictured in Ref. 7. Instead the oscillations would suddenly disappear. After these experiments we also took the opportunity to look at *subharmonic resonances* at the oscillator frequencies ω_0/n [Fig. 5(c)]; these are simply overtones produced in the nonlinear element coinciding with the resonance frequency.

It is obvious that the circuit experiment gave incomparably the best opportunities to study the parametric effects from the quantitative side. At the same time, there is a danger in this since the students quite soon forget what is going on in the elements and concentrate on the oscilloscope traces. We therefore finished by repeating the energy and phase argument: at which points during the resonant oscillation should the capacitance be increased and decreased in order that work be done on the system?

Modern students are apt to ask about the usefulness of the investigated effects. We therefore concluded by briefly describing parametric amplifiers and the enormous importance of parametric effects in nonlinear optics, plasma physics, etc.

VI. CONCLUSION

A rather complete description has been given of some experiments on parametric oscillations. Although we have gone into some analytical detail in the case of the resonance circuit, the experiments were designed to be constructed by the students with only a qualitative picture of the theory. In this picture the oscillator gains energy for a certain phase relation to the parameter variations. The emphasis was on building up the system to discover a presumably unknown effect. Actually, we discovered an interesting recurrence phenomena in the spring pendulum.

ACKNOWLEDGMENTS

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