Trajectories of projectiles in air for small times of flight

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Angle of departure, angle of fall, and the position of the summit of trajectories in air are calculated with neglect of the effect of the air on the fall from the line of departure. Experimental information required is times of flight against range or remaining velocities against range. Calculated values are compared with observed values for a supersonic bullet.

I. INTRODUCTION

The paths of bullets and other projectiles in air differ from their paths in a vacuum. For a given range and initial velocity, the angle of departure and the maximum height reached are larger in air. Moreover, the path is no longer parabolic: the maximum height of the trajectory for bullets is closer to the target than to the shooter and the angle of fall is steeper than the angle of departure. Nevertheless, realistic trajectories can be calculated quite simply for horizontal ranges provided that the time of flight is no more than a second or so. This condition is satisfied for the ranges normally used for aimed shots during rifle shooting.

II. NEGLECT OF AIR RESISTANCE FOR VERTICAL DISPLACEMENTS

The displacement of a projectile at any time is the result of two components: a displacement along the line of departure and a vertical drop from the line of departure. The first displacement is affected by air resistance but not by gravity. The second is affected by both air resistance and gravity but, provided the time of flight is less than a second or so and the angle of departure less than a degree or so, the effect of air resistance can be neglected for this vertical drop. Figure 1 shows the two displacements for the *special case* of a projectile that has returned to the level of the point of departure.

The magnitude s of the displacement along the line of departure is given by

$$s = \overline{V}t, \tag{1}$$

where $\overline{V}$ is the mean velocity along the line of departure and t is the time after projection. (In the absence of air resistance $\overline{V}$ would always equal the initial velocity.)

For short periods of time, the fall h *from the line of departure* is given by

$$h = \tfrac{1}{2}gt^2. \tag{2}$$

This applies for the drop to any point on the trajectory. Although Eqs. (1) and (2) can be used directly to plot a trajectory, it is more satisfactory to use them and other information to obtain equations for the angle of departure, position of the summit, and angle of fall. Experimental data sufficient to obtain a trajectory is then a set of times of flight for various horizontal distances, as shown in Fig. 2. These experimental data and others in this article are taken from Whelen's book on ballistics.[1] As an alternative to times of flight, we can begin with remaining velocities against distance. From this times of flight can be calculated from $\int 1/v\, ds$.

III. ANGLE OF DEPARTURE AND POSITION OF SUMMIT

The angle of departure for a given range can be calculated for a bullet in air from the time of flight only. Refer again to Fig. 1. The bullet leaves S at an angle of departure α to the horizontal and follows a curved trajectory to the target at T. The displacement along the line of departure is SH; its magnitude is s. The vertical displacement is HT; its magnitude is h. For a projectile that has just hit point T the time taken for the displacement SH equals the time for the vertical displacement HT. This is the time of flight t. Provided that t is small h is given by Eq. (2). Now $h = R \tan\alpha$ so

$$\tan\alpha = gt^2/2R, \tag{3}$$

where R is the range.

This neglects air resistance for the fall HT but not for the displacement SH.

The value of t in Eq. (3) is obtained from a plot of R against t as in Fig. 2.

Provided that the angle of departure is small we can ignore the effect of air resistance on the vertical component of the velocity of a projectile since this component will be small. The time taken to fall from the summit of the trajectory will then be half the total time of flight. Since the motion at the summit is purely horizontal,

$$y_{\max} = \tfrac{1}{2}g(t/2)^2 = gt^2/8, \tag{4}$$

where $y_{\max}$ is the maximum height of the curved trajectory and t the *total* time of flight. It is emphasized that $y_{\max}$ is the actual height of the summit and is not equal to h, which is the notional fall from the line of departure.

Because the projectile reaches the summit in half the time of flight, its horizontal distance x_s from the start is given by

$$x_s = R \text{ for } t/2. \tag{5}$$

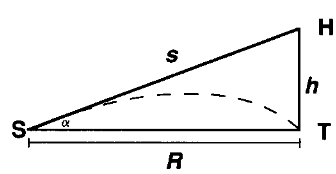

Fig. 1. A projectile leaves point S and strikes point T on the same level. ST is the range, and α is the angle of departure. The range is the resultant of two displacements, one of magnitude s along the line of departure OH and one of magnitude h along HT where HT is the fall from the line of departure. For small times of flight, air resistance can be neglected for the vertical motion.

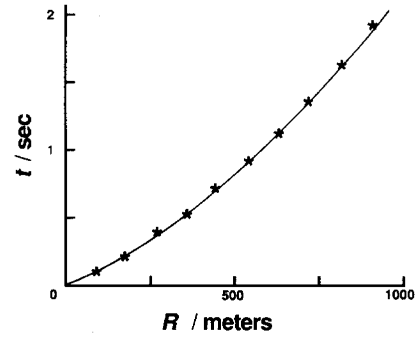

Fig. 2. Experimental values for times of flight for a pointed bullet of mass 9.7 g, calibre 0.76 cm, and muzzle velocity 823 ms^{-1}. Raw results are taken from Whelen.

The value of R for $t/2$ can be found from a graph of times versus distance (Fig. 2).

IV. ANGLE OF FALL

At the target, the velocity can be regarded as the result of a vertical and a horizontal component. The horizontal component is $V_R \cos \alpha_f$, where V_R is the remaining velocity and α_f is the angle of fall. Provided α_f is small, which it is for normal ranges, the horizontal component can be taken as equal to V_R in magnitude. We ignore air resistance for the vertical drop from the summit, so the vertical component of the velocity at the target equals $gt/2$, where t is the total time of flight. The angle of fall is then given by

$$\tan \alpha_f = gt/2V_R . \quad (6)$$

The remaining velocity at the target can be obtained from the plot of range against time since $V = dR/dt$. Nevertheless, observed figures for remaining velocities against distance are often available,[1] so this may be unnecessary.

V. RATE OF CHANGE OF ANGLE OF DEPARTURE

The absolute angle of departure is of limited practical value. This is because it does not, in general, equal the angle of elevation of the bore of the weapon just before firing.[2]

For a vacuum, $d\alpha/dR$ is constant but, in air, it increases with range. The smaller $d\alpha/dR$ the better. Indeed, a zero value would be ideal since no sight adjustment would be needed for varying ranges. The increase in $d\alpha/dR$ resulting from air resistance is a major inconvenience in marksmanship. This derivative can be obtained geometrically from a plot of α against R. The plot can be obtained from Eq. (3) and experimental times of flight as in Fig. 2.

VI. TRAJECTORIES FROM DRAG COEFFICIENTS

The deceleration $-a$ of a projectile due to air resistance is given by

$$a = -kv^2 , \quad (7)$$

where

$$k = iK_D \rho d^2/m , \quad (8)$$

also expressed as

$$k = \Pi/8C_D \rho d^2/m , \quad (9)$$

where ρ is the density of air, d is the diameter, v is the velocity, and m is the mass of the projectile. The coefficients k, C_D, and iK_D are not constants for a given projectile. Near the speed of sound, as the projectile loses speed, there is a sharp drop in their values. At higher and lower speeds, they are approximately constant over a range of velocities for a given projectile.[3–5]

For a given projectile in air at constant density, it is appropriate to use Eq. (7). The rate of change of velocity with distance x is of more relevance than acceleration, so we

Table I. Calculated and observed parameters for trajectories. Angle of departure, angle of fall, summit height, and position of the summit for trajectories of different ranges for the same projectile. The calculated results were found from the time of flight and Eqs. (3)–(6). The observed results were taken from trajectories obtained directly from shooting. The times of flight and the measured characteristics were provided by Whelen. The projectile was mass 9.7 g, diameter 0.76 cm, and had a muzzle velocity of 823 ms^{-1}.

Range/m	91	183	274	366	457	549	640	732	823	914
Summit height/m										
(a) Calculated	0.01	0.05	0.20	0.30	0.60	1.1	1.5	2.1	3.1	4.4
(b) Observed	0.02	0.1	0.19	0.33	0.61	1.0	1.5	2.1	3.3	4.6
Position of summit/m										
(a) Calculated	48	100	160	200	270	320	360	440	510	560
(b) Observed	48	99	147	196	246	299	357	416	478	540
Angle of departure/m										
(a) Calculated	1.9	3.7	9.8	12	18	25	32	39	52	66
(b) Observed	2	5	8	12	16	20	26	33	40	49
Angle of fall/m										
(a) Calculated	2.2	5	11	15	23	34	46	61	83	106
(b) Observed	2	6	10	15	22	31	42	57	76	98

write

$$a = \frac{dv}{dt} = \frac{dv}{dx}\frac{dx}{dt} = v\frac{dv}{dx}. \quad (10)$$

Therefore, from Eqs. (7) and (10)

$$\frac{dv}{dx} = -kv. \quad (11)$$

This is valid only if k is independent of velocity and distance traveled x, at least approximately.

From Eq. (11)

$$\int \frac{dv}{v} = \int -k\,dx. \quad (12)$$

Therefore,

$$\ln v_2/v_1 = -k(x_2 - x_1). \quad (13)$$

A special case is

$$\ln v/v_0 = -kR, \quad (14)$$

where v_0 is the velocity at the point of departure and R is the horizontal distance traveled. For the small angles of departure used for small arms, R can be taken as equal to the total curved distance x.

From Eq. (14)

$$v = v_0 \exp(-kR). \quad (15)$$

Now

$$t = \int (1/v)dR$$

so

$$t = (1/kv_0)(e^{kR} - 1). \quad (16)$$

Also

$$t = (1/k)(1/v - 1/v_0). \quad (17)$$

From Eqs. (3) and (16), the angle of departure is given by

$$\tan\alpha = g/2Rk^2v_0^2(e^{kR} - 1)^2. \quad (18)$$

From Eqs. (6), (15), and (16) the angle of fall is given by

$$\tan\alpha_f = ge^{kR}(e^{kR} - 1)/2kv_0^2. \quad (19)$$

From Eqs. (3) and (4) the maximum height of the trajectory is given by

$$y_{\max} = (R\tan\alpha)/4. \quad (20)$$

Equation (17) is useful for calculating times of flight from a set of remaining velocities and corresponding ranges; the mean value of k is found by way of Eq. (13).

Equations (3), (4), and (6) have the advantage of simplicity over Eqs. (18), (19), and (20). Also, for the most accurate work, they require no more data. This is because a set of results of time of flight versus distance, or remaining velocities versus distance, is required to obtain a reliable value of k. However, Eqs. (18), (19), and (20) are useful for rough calculations because iK_D is often between 0.1 and 0.2 for bullets with a sharp point and flat base traveling above the speed of sound and also for round nose bullets below the speed of sound. A representative value is

$$iK_D = 0.15.$$

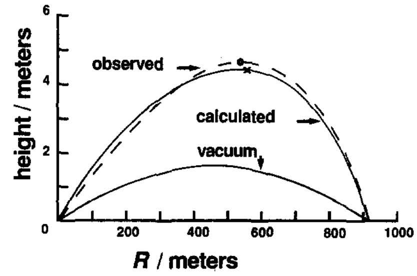

Fig. 3. A vacuum trajectory and the observed trajectory compared with a trajectory calculated from the method described in the text. The latter two curves are sketched from values of angle of departure, angle of fall, and position of the summit. Agreement improves with decreasing range.

VII. COMPARISON OF CALCULATED AND OBSERVED VALUES

Comparison of calculated and observed values of angles of departure, angles of fall, height of the summit, and position of the summit is made in Table I. These values are for the projectile of Fig. 2, which has a muzzle velocity of 823 ms^{-1}. The calculated values come from experimentally determined times of flight extracted from Whelen[1] and Eqs. (3)–(6). The observed values of the features of the trajectories are also extracted from Whelen.

In plotting trajectories on normal-sized graph paper, it is necessary to go to very long ranges before the calculated trajectories and observed trajectories are noticeably different. The two trajectories for 914 m are shown in Fig. 3 together with the vacuum trajectory, which is a long way out.

Rates of change of angle of departure with range for the same projectile and muzzle velocity are shown in Table II. The calculated values of $\Delta\alpha/\Delta R$ were found geometrically from the plot of α against R; this plot was obtained from Fig. 2 by way of Eq. (3). The observed values, obtained directly by shooting, are taken from Whelen.[1] Agreement

Table II. Rates of change of angle of departure. The rate of change of angle of departure with range for a sharp-nosed bullet of mass 9.8 g, diameter 0.76 cm, and initial velocity of 823 ms^{-1}. The calculated values come from the times of flight. The observed values come directly from shooting. For a vacuum, the calculated value is 2.5 min per 100 m.

	(min per 100 m)							
	100–200 m	200–300 m	300–400 m	400–500 m	500–600 m	600–700 m	700–800 m	800–900 m
Calculated	3.0	3.8	5.0	6.0	7.0	9.0	13.5	17
Observed	3.0	3.5	4	5	5.5	6.5	9.0	12

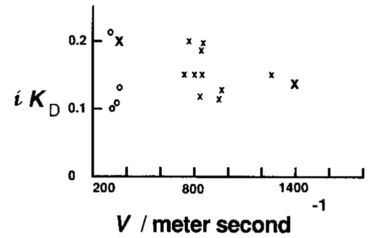

Fig. 4. The coefficient of drag iK_D plotted against velocity for projectiles of different masses and calibres. ○ indicates round-nosed bullets and × sharp-nosed. These results were calculated from remaining velocities versus times of flight for small arms taken from Whelen. The large X is for a 10-cm artillery shell. The coefficient was taken from the Encyclopaedia Brittanica.

is good up to about 400 m, since random errors in shooting commonly exceed 1 min of angle. For longer ranges, the errors are significant. Nevertheless, the calculated values are far more realistic than the constant value of 2.5 min per 100 m calculated for a vacuum.

The usefulness of Eqs. (15)–(19) depends on how constant is the coefficient iK_D (or C_D). Figure 4 shows the variation of iK_D with velocity for projectiles of different velocities. The value of iK_D also varies with distance covered for a given projectile and muzzle velocity; for example, for the projectile of Fig. 2, iK_D varied between 0.13 and 0.18 over a range of 900 m. Nevertheless, the mean figure of 0.15 taken with the density of air as 1.2 kg m^{-3} gave good results up to 400 m when used with Eqs. (8) and (15)–(19), and realistic results up to the very long range of 914 m. Agreement between calculated and observed results was similar for a sharp-nosed bullet of mass 11.2 g, calibre 0.79 cm, and muzzle velocity 744 ms^{-1}, and also for a round-nosed bullet of mass 26.1 g, calibre 11.4 cm, and muzzle velocity 350 ms^{-1} when the value of iK_D was taken again as 0.15.

Taking the value as 0.15 for all bullets could give substantial error at times. Nevertheless, it is better than ignoring air resistance altogether, which amounts to taking iK_D and k as zero.

The treatment given in this article is not valid for calculating trajectories for maximum ranges. This is because air resistance cannot be ignored for the vertical motion when the time of flight exceeds a second or so. For the projectile of Fig. 2, the observed maximum range of 3156 m was obtained at an angle of departure of approximately 30° and the time of flight was 26 s. However, Eq. (3) gives a value of 46° for this range.

VIII. SUMMARY

Completely ignoring air resistance for bullets in air gives trajectories that are unrealistic. Ignoring air resistance only for the vertical component of the motion is an improvement over neglecting it entirely and gives realistic trajectories for the ranges normally used for small arms—up to about 400 m for hunting and to about 1000 m for military purposes.

In calculating the angle of departure for a given horizontal range, the total displacement is considered to have two components: a notional displacement along the line of departure subject to air resistance and a notional vertical displacement from the line of departure for which air resistance is neglected. This gives Eq. (3). The height of the summit is calculated by neglecting air resistance for the actual vertical displacement, so the time to reach the summit is half the total time to reach the target. This gives Eq. (4). The horizontal position of the summit is found from the distance traveled in half the time of flight.

The angle of fall is calculated by neglecting the effect of air resistance on the vertical component of the velocity and by neglecting the difference in magnitude of the remaining velocity and its horizontal component. This gives Eq. (6) and also Eq. (5) for the position of the summit.

The approximations behind Eqs. (3)–(6) imply small angles of departure (1° or so) and small times of flight (1 s or so). The data required are a set of times of flight against range or a set of remaining velocities against range. This method is much simpler than the approach through drag coefficients and requires no more experimental data. However, the drag coefficient approach throws light on factors that affect air resistance. Also, Eq. (17) provides a more convenient way of calculating times of flight from remaining velocities than counting squares under the curve of $1/v$ versus distance.

Although no equation is given here for the path of a projectile in air, easily derived relationships are provided which give useful nonparabolic trajectories in air. A previous paper[2] gives a semiempirical equation for curves that are a good fit to observed trajectories in air, but being parabolic it does not give correct values for angles of departure, angle of fall, and position of the summit.

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