# An equation for the caustic curve


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In his book *Experiments in Physics* (Siddons 1988), Colin Siddons speaks fondly of the caustic curve and mentions that he has yet to discover an equation for the cusp. That remark is the motivation behind this paper for I would like to advance a possible solution.

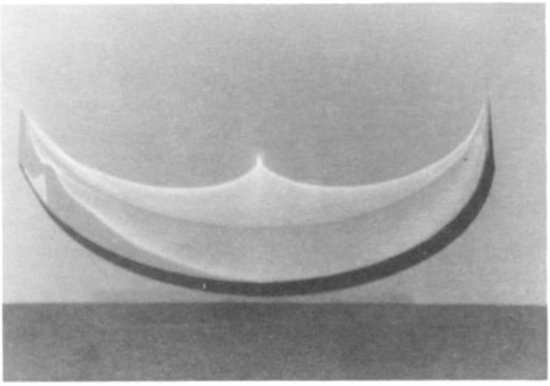
### The caustic curve

In the analysis which follows I will be considering the caustic curve formed when a collimated beam of light is aligned along the principal axis of, and incident upon, a semicircular cylindrical mirror such that the beam completely fills the aperture of the mirror. An example of a caustic curve so formed is shown in figure 1.

Such a mirror suffers from spherical abberation; rays reflected from different parts of the mirror are brought to a focus at different places. This may be illustrated by a delightful experiment which can be found in the *Nuffield Physics Teacher's Guide* (Rogers and Wenham 1977). Briefly, a comb is placed in front of a lamp and the resulting fan of rays fills the aperture of a cylindrical mirror. A card containing a slot is moved across the comb so that only a few rays are reflected from the mirror at a time. As the card is moved across the comb so the image slides around the caustic curve (figure 2).

By constructing a caustic curve from a ray diagram (such as in figure 3) an important point emerges with regard to the argument that follows: it may be seen that the reflected rays are the tangents to the caustic curve. The diagram was drawn by sending in rays parallel to the principal axis, constructing a normal where the ray strikes the mirror and finally obeying the law of reflection at that point.

**Figure 1.** Caustic curve produced with a semicircular cylindrical mirror.



### Finding an equation for the reflected ray

Consider a semicircular cylindrical mirror whose radius of curvature is $R$. The geometry of the situation that I wish to use is shown in figure 4. The incident ray LM is parallel to the principal axis and at a distance $a$ from it. Let $\angle LOM = \alpha$. Construct the line ON to be perpendicular to the reflected ray MQ. Now, $\angle LMO = \angle OMN$ by the law of reflection and $\angle OLM$ and $\angle ONM$ are both right angles. Since OM is the common hypotenuse

**Figure 2.** Mirror and comb experiment.

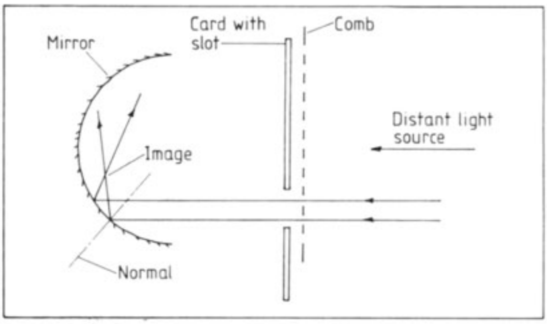
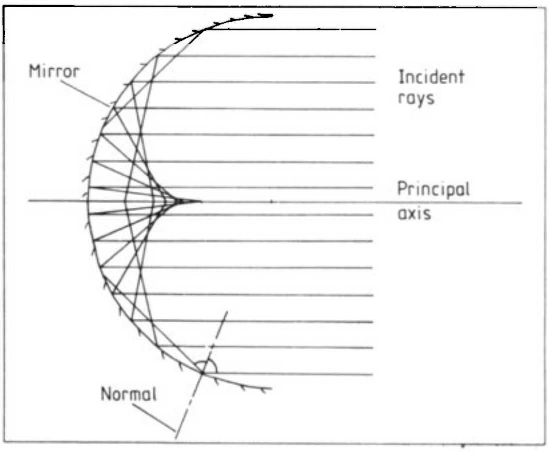



**Figure 3.** The reflected rays are tangents to the caustic curve.



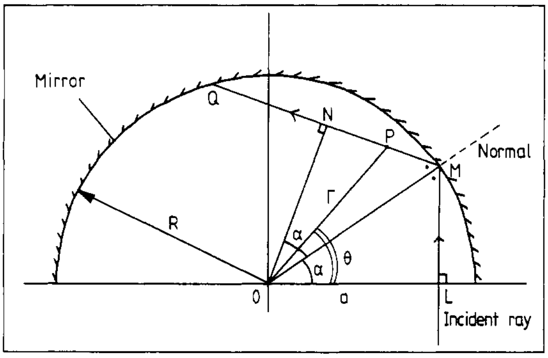



**Figure 4.** Geometry needed to find the equation for the reflected ray.

it follows that triangles OML and OMN are congruent. Hence ∠NOM = $\alpha$ and ON = $a$.

Choose an arbitrary point P on the reflected ray. Let the line OP make an angle $\theta$ with OL and let P be a distance $r$ from O.

Now consider triangle NOP. From the diagram it is seen that ∠NOP = $(2\alpha - \theta)$. Hence

$$\cos(2\alpha - \theta) = a/r$$

or

$$r = a \sec(2\alpha - \theta).$$

But from triangle LOM

$$\cos\alpha = a/R$$

or

$$a = R\cos\alpha.$$

Hence the polar equation to the reflected ray is

$$r = R\cos\alpha \sec(2\alpha - \theta). \qquad (1)$$

$\alpha$ (and so $a$) is an adjustable parameter. If we sweep through all the values that $\alpha$ can take (from 0 to $\pi$) then we have described all the possible singly reflected rays in our arrangement.

### From the reflected rays to the caustic curve

Define $F$ such that

$$F = r - R\cos\alpha \sec(2\alpha - \theta).$$

As mentioned earlier, the caustic curve is the common tangent to the reflected rays. It is said to be the envelope of those rays and may be found by solving the two equations

$$F = 0 \qquad \text{and} \qquad \frac{\partial F}{\partial \alpha} = 0$$

(Aroian *et al* 1968). We have the first of these as (1) above. If we take the partial derivative of $F$ with respect to $\alpha$ we obtain

$$\frac{\partial F}{\partial \alpha} = R\cos\alpha \sec(2\alpha - \theta)(\tan\alpha - 2\tan(2\alpha - \theta)).$$

But we must set $\partial F/\partial\alpha = 0$ which leads to three options:

(i) $\sec(2\alpha - 0) = 0$ which has no solution;

(ii) $\cos\alpha = 0$ which implies $\alpha = \pi/2$. This corresponds to a ray incident and reflected along the principal axis; and

(iii) $\tan\alpha - 2\tan(2\alpha - \theta) = 0$. This is the interesting one.

To solve for $\alpha$ we must first use the trigonometrical identities

$$\tan(A - B) = \frac{\tan A - \tan B}{1 + \tan A \tan B}$$

and

$$\tan 2A = \frac{2\tan A}{1 - \tan^2 A}$$

to obtain

$$\tan^3\alpha + 3\tan\alpha - 2\tan\theta = 0.$$

$\tan\alpha$ may be found from this cubic by using Cardan's formula (Aroian *et al* 1968). (This is a standard mathematical procedure akin to using Al Khwarismi's familiar rule for finding the roots of quadratic equations.) Application of this technique yields

$$\tan\alpha = (\sec\theta + \tan\theta)^{1/3} - (\sec\theta - \tan\theta)^{1/3}.$$

We can now combine this solution with (1) above.

### The equation to the caustic curve

If a caustic curve is formed by a collimated beam of light being reflected from a semicircular cylindrical mirror of radius $R$ (such that the beam fills the aperture of the mirror and is directed along the principal axis) then its equation is

$$r = R\cos\alpha \sec(2\alpha - \theta)$$

where

$$\alpha = \tan^{-1}[(\sec\theta + \tan\theta)^{1/3} - (\sec\theta - \tan\theta)^{1/3}].$$

For the convenience of those who wish to plot a caustic curve, the length of the polar radius $r$ for angles between $\theta = 0$ and $\theta = 90°$ at 5° intervals is given in table 1. As the coordinates have been calculated for a mirror of unit radius, $r$ may simply be scaled up or down to suit any situation.

Table 1. Polar coordinates for the caustic curve.

| θ (deg) | $r$ |
|---|---|
| 0 | 1.0000 |
| 5 | 0.9987 |
| 10 | 0.9949 |
| 15 | 0.9886 |
| 20 | 0.9797 |
| 25 | 0.9682 |
| 30 | 0.9542 |
| 35 | 0.9375 |
| 40 | 0.9183 |
| 45 | 0.8963 |
| 50 | 0.8716 |
| 55 | 0.8441 |
| 60 | 0.8135 |
| 65 | 0.7796 |
| 70 | 0.7421 |
| 75 | 0.7001 |
| 80 | 0.6524 |
| 85 | 0.5953 |
| 90 | 0.5000 |

**Acknowledgments**

I would first like to thank Mr J C Siddons for valiantly hacking his way through my mathematical jungles when I communicated my results. Secondly my thanks must go to both Neil Brown and Anthony Hamilton for their work in providing the photograph which accompanies this article.

This article is dedicated to my father, John McIntosh.

**References**

Aroian L A *et al* (ed) 1968 *Van Nostrand's Scientific Encyclopaedia* (New York: Van Nostrand) (See entries under 'Envelope' p 627 and 'Cubic Equation' p 469)

Rogers E M and Wenham E J (ed) 1977 *Revised Nuffield Physics Teacher's Guide Year 3* (London: Longman) p 69 (class expt 19L)

Siddons J C 1988 *Experiments in Physics* (Oxford: Blackwell) pp 53–4

# An experiment to measure the latent heat of vaporisation of liquid nitrogen

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The experiment consists of the measurement of the latent heat of vaporisation of liquid nitrogen at 77.3 K, obtained from the volume of gas evaporated when the liquid is fed with a known amount of energy provided by a constant rate of electric power dissipated during the Joule effect in an electrical resistor.

The pressure variation during the experiment due to the particular method chosen to measure gas volumes introduces a rather surprising and interesting thermodynamic effect which depends upon the amount of liquid nitrogen used.

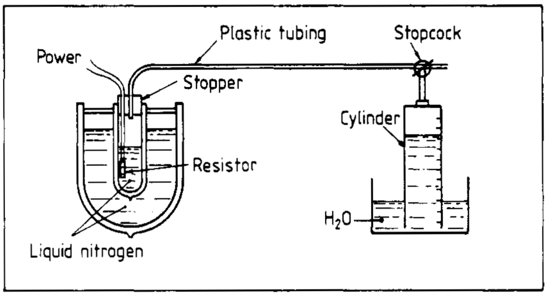


Figure 1. Schematic of the calorimeter.

### Description of the device

As shown in figure 1, the calorimeter is composed of two coaxial Dewar vessels containing liquid nitrogen. An electrical resistor, whose connections pass through a rubber stopper in the neck of the inner vessel, is fully immersed in the liquid.

The evaporated nitrogen is tightly transferred to