Using spreadsheet modelling to teach about feedback in physics

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Abstract
This article looks generally at spreadsheet modelling of feedback situations. It has several benefits as a teaching tool. Additionally, a consideration of the limitations of calculating at many discrete points can lead, at A-level, to an appreciation of the need for the calculus. Feedback situations can be used to introduce the idea of differential equations. Microsoft Excel™ is the spreadsheet used.

Introduction

Situations involving feedback abound in physics. They are commonly expressed mathematically as differential equations in which the rate of change of a quantity is a function of that quantity. They can occur in situations involving feedback loops (both positive and negative feedback). Two examples are as follows:

- radioactive decay ($dN/dt = -\lambda N$, where N is the number of nuclei present and λ is the probability per unit time of decay, known as the decay constant). Here the differential equation represents the feedback loop:

   *"the activity (dN/dt) causes
   a decrease in N, which causes
   a decrease in activity, which causes
   a smaller decrease in N, which
   causes…"*

- a body falling in the presence of a drag force (Newton's second law gives $m\Delta v = (mg - kv^2)\Delta t$, where v is the velocity of an object of mass m, k is a 'drag coefficient' and kv^2 is the drag force). The feedback loop corresponding to this is the familiar terminal velocity case—see the Frontline article on page 383 of this issue.

Severn (1999) uses spreadsheet modelling to provide solutions to such differential equations.

For example, by using suitably small time intervals, the equation $m\Delta v = (mg - kv^2)\Delta t$ can be used to calculate successive values of v over time. The treatment of terminal velocity also uses iterative processes on a spreadsheet, but I use them *at each stage in the feedback loop*, instead of on the fully formed differential equation. This confers the following four benefits:

- younger students who have no understanding of differential equations can produce solutions to them, using very basic physics (in the terminal velocity example, very little more than $F = ma$ is required).
- the variation with time of all parameters in the equation can be investigated (in the terminal velocity example acceleration–time graphs can be formed as well as velocity–time graphs).
- causal relationships and the nature of the feedback loop become apparent from the spreadsheet in a more explicit way than from a fully formed differential equation.
- the spreadsheet shows up the shortcomings of the numerical modelling more explicitly, which helps to explain the usefulness of the calculus.

I will discuss each benefit in turn, and illustrate each with examples of spreadsheet modelling that can be performed by students in the classroom.

	A	B	C	D	E	F	
1			**Discharge of a capacitor through a resistor**	
2							
3	0.001		t / s	V / V	I / A	Q / C	
4	capacitance (cap)		0	=V0	=D4/res	=cap*V0	
5			=C4+1	=F5/cap	=D5/res	=F4-E4	
6			=C5+1	=F6/cap	=D6/res	=F5-E5	
7	10000		=C6+1	=F7/cap	=D7/res	=F6-E6	
8	resistance (res)		=C7+1	=F8/cap	=D8/res	=F7-E7	
9			=C8+1	=F9/cap	=D9/res	=F8-E8	
10	9		=C9+1	=F10/cap	=D10/res	=F9-E9	
11	V_0		=C10+1	=F11/cap	=D11/res	=F10-E10	
12			=C11+1	=F12/cap	=D12/res	=F11-E11	

Figure 1. The first few lines of formulae modelling the discharge of a 1000 μF capacitor through a 10 kΩ resistor.

	A	B	C	D	E	F	
1			**Discharge of a capacitor through a resistor**	
2							
3	0.001		t / s	V / V	I / A	Q / C	
4	capacitance (cap)		0	9.000	0.000900	0.00900	
5			1	8.100	0.000810	0.00810	
6			2	7.290	0.000729	0.00729	
7	10000		3	6.561	0.000656	0.00656	
8	resistance (res)		4	5.905	0.000590	0.00590	
9			5	5.314	0.000531	0.00531	
10	9		6	4.783	0.000478	0.00478	
11	V_0		7	4.305	0.000430	0.00430	
12			8	3.874	0.000387	0.00387	

Figure 2. The first few lines of calculated data modelling the discharge of a 1000 μF capacitor through a 10 kΩ resistor.

Younger students gain access to physics that would usually be beyond them mathematically

Applying $F = ma$ to the case of a falling body in air gives $m\Delta v = (mg - kv^2)\Delta t$, where mg is the weight and kv^2 is the drag. Therefore $mg - kv^2$ is simply the resultant force, and acceleration has been written as $\Delta v/\Delta t$.

A typical 14-year-old in the UK would not be expected to solve an equation such as this, and yet we have seen in the Frontline article that (s)he can perform sophisticated quantitative work using spreadsheet modelling and produce what amounts to a graphical solution of this equation.

More than one variable can be modelled at a time

In the terminal velocity example above:

"the velocity is calculated from the acceleration, which comes from the resultant force, which comes from the drag, which comes from the velocity …"

This means that the time dependence of any of these quantities can be observed immediately

from the appropriate column of the table, or by constructing a separate graph. We have already seen that variations in the acceleration can be linked to the changing gradient of the velocity–time graph.

Another example is the modelling of capacitor discharge through a resistor. The spreadsheet given is for a 1000 μF capacitor and a 10 kΩ resistor. Figures 1 and 2 show the first few lines of formulae and calculated data, respectively, while figure 3 is the graph showing the decay of potential difference across the capacitor.

The spreadsheet is easy for A-level students to construct, because the only physics required are the formulae $Q = CV$ and $V = IR$. The exponential discharge formula $Q = Q_0 \, e^{-t/CR}$ is not required, and thus students can appreciate where exponential relationships in physics come from. The trick (which makes the feedback appear) is to realize that (since the time intervals, Δt, are equal to 1 s and $I = \Delta Q/\Delta t$) each charge value is equal to *the previous one minus the appropriate current*.

The p.d. against time graph is shown in figure 3 but it is equally straightforward to
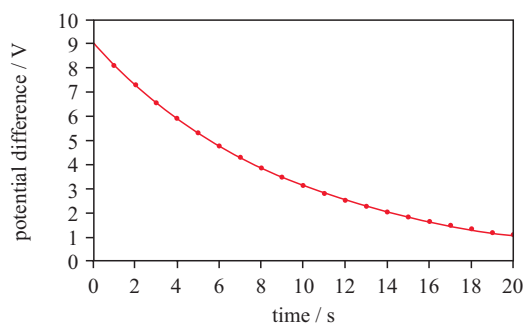
Figure 3. The decay of potential difference across a capacitor.

construct graphs of charge stored against time, or current against time. In this way, the student can appreciate that all three of V, I and Q decay exponentially.

A affects B affects C affects A...

Because each variable has a column of its own in the spreadsheet, and each column is calculated from at least one other, it is easy to see the causal relationships within the feedback loops. We have already seen that, in the terminal velocity example:

> *"the velocity is calculated from the acceleration column, which comes from the resultant force column, which comes from the drag column, which comes from the velocity column..."*

Many students will gain more insight from this into the physics of the situation than they will from forming a single differential equation and then solving it.

I like my students to express these relationships diagrammatically, using arrows on a blank copy of the spreadsheet. For example, if A → B means 'B is calculated from A', then the iterative calculations on the terminal velocity spreadsheet could be represented as shown in figure 4. This diagrammatic representation makes it easy to see which variables affect which others, and the repeating pattern makes clear the nature of the feedback loop.

The feedback loop that leads to the exponential discharge of a capacitor can be written as:

> *"The initial potential difference determines the current, which changes the charge stored, which changes the potential difference, which changes..."*

Figure 4. Representation of the calculations on the terminal velocity spreadsheet.

Figure 5. Representation of the exponential discharge of a capacitor.

The diagrammatic representation of this feedback loop is shown in figure 5.

Note that the differential equations modelled in this article all deal with rates of change with respect to time, but this modelling technique will work for all feedback situations. The easier the physics at each step of the feedback loop, the more accessible the situation will be to students. Complicated overall pictures (see, for example, $m\Delta v = (mg - kv^2)\Delta t$, which *is* complicated if you are a British 16-year-old!) can be broken down into a series of easy steps. In my opinion, once the student is used to working with spreadsheets, the modelling is then only as hard as the hardest single step.

All the models in this article have in common that the *next value* is calculated by adding (or subtracting) a rate of change to a *previous value*. For example,

$$v(t) = v(t - 1) + acceleration$$

in the model of terminal velocity,

$$Q(t) = Q(t - 1) - current$$

in the capacitor discharge model.

If the time intervals are not equal to 1, then the relevant fraction or multiple of a rate of change

	A	B	C	D	E
1			**Radioactive decay**		
2					
3	decay constant (lambda)		time	Activity	Number
4	0.4		(s)		remaining
5	initial number (N-zero)		0	=INT(lambda*E5)	=N_zero
6	5000000		=C5+1	=INT(lambda*E6)	=E5-D5
7			=C6+1	=INT(lambda*E7)	=E6-D6
8			=C7+1	=INT(lambda*E8)	=E7-D7
9			=C8+1	=INT(lambda*E9)	=E8-D8

Figure 6. The first few lines of formulae for the decay of 5 million atoms with $\lambda = 0.2$ s^{-1}.

	A	B	C	D	E
1			**Radioactive decay**		
2					
3	decay constant (lambda)		time	Activity	Number
4	0.2		(s)		remaining
5	initial number (N-zero)		0	1000000	5000000
6	5000000		1	800000	4000000
7			2	640000	3200000
8			3	512000	2560000
9			4	409600	2048000

Figure 7. The first few lines of calculated data for the decay of 5 million atoms with $\lambda = 0.2$ s^{-1}.

will be needed. More advanced students may well see the connection between these relationships and the calculus, but unfortunately this has not yet happened in my classes!

Models can be mathematically incorrect!

In the terminology of the previous section, we can set up a model of radioactive decay (in fact, it is the most straightforward of the three models presented here) by using the number of nuclei to calculate the activity to calculate the number of nuclei to calculate the activity... etc, i.e.

$$N(t) = N(t-1) - A(t-1)$$

and $A(t) = \lambda N(t)$ where $A = \mathrm{d}N/\mathrm{d}t$ is the activity of the sample.

Figures 6, 7 and 8 show the spreadsheets and graph for the radioactive decay of 5 million atoms of an isotope whose decay constant, λ, is equal to 0.2 s^{-1}. In other words, each nucleus has a 1 in 5 chance of decaying each second. The INT function has been used to round off numbers to integer values, in recognition of the fact that fractions of an atom are meaningless.

Notice that the number of nuclei and the activity can both be modelled on the same graph, and that the nature of the feedback is clear from the table. Diagrammatically the feedback loop looks as shown in figure 9. The simplicity of
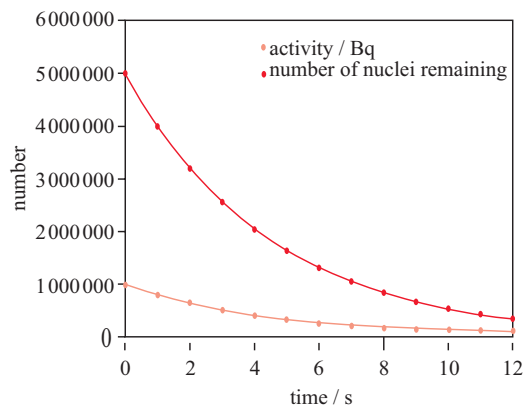

Figure 8. The radioactive decay of 5 million atoms with $\lambda = 0.2$ s^{-1}.

the feedback diagram reflects the comparative simplicity of setting up this particular spreadsheet.

At first sight, it looks like the graph shows a nice exponential function. We are told that exponential functions result when the rate of change of a quantity is proportional to that quantity, and it looks as though that is the case here, since the activity at each time is found by multiplying the number of nuclei by a constant fraction (the decay constant).

However, by drawing construction lines on the graph it can be seen that the half-life increases with time, which should not happen! In fact the curve is not truly exponential at all. This can

time / s	number of nuclei	activity / s^{-1}
0		
1		
2		
3		

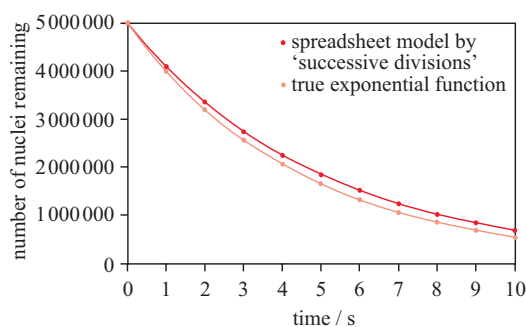**Figure 9.** Feedback diagram for radioactive decay.

Figure 10. Comparison of the original model and the true exponential function.

be seen in figure 10, in which the original model is plotted alongside the true exponential function $N = N_0 e^{-0.2t}$. The model has values of number of nuclei remaining that are too low (another way of saying this is that the model overestimates the activity).

Why is this? Why does our model not provide a true exponential function? Well, our model calculates an activity from the number of nuclei, and it does it very well, but it applies this activity *for the whole second*, until the next row of the spreadsheet, when it recalculates. In reality, the activity changes not in discrete steps, but continually. In each interval of one second the activity will be falling, whereas our model assumes it to stay at its initial value. This leads our model to overestimate the activity at each step, and to underestimate the number of nuclei remaining. (As an aside, for this reason those of you who throw 'Anderson cubes' to model radioactive decay should expect a better fit to our spreadsheet model than to the true exponential function!)

The student should see easily from this that using smaller time intervals, δt, should give a

better model, with a better fit to 'reality'. This can easily be done on the spreadsheet by the student: using time intervals of, say, $\delta t = 0.1$ s does indeed take the modelled line closer to the true exponential function. The implication is that the true function will be reached in the limit $\delta t \to 0$. This exercise thus shows the student that a model is just that—only a model, and it can help students to see why the calculus is so useful and how it comes about.

The difference between the model and the true exponential function can be seen mathematically. Each value of N in the model is just four-fifths ($1 - \lambda$) of the previous value. Thus $N = N_0(1 - \lambda)^t$. A-level students with good mathematical skills could be given the task of expanding this function binomially and comparing it with a power series expansion of $N = N_0 e^{-\lambda t}$.

Conclusion

Spreadsheets can be used to model physical situations involving feedback loops, and thus provide graphical solutions to differential equations. Parameters can easily be changed and their effects investigated. The causal relationships that cause the feedback loop become apparent in the spreadsheet model and it can be instructive for the student to make them explicit in diagrammatic form. The student can benefit from considering the mathematical limitations of the model. Spreadsheet modelling can help the student understand feedback situations without having to learn differential equations, and therefore the student can learn certain areas of physics quantitatively at a younger age than would otherwise be the case.

Reference

Severn J 1999 Use of spreadsheets for demonstrating the solutions of simple differential equations *Phys. Educ.* **34** 360–6

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