Slowly varying envelope approximation in a laser with optical feedback

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We examine the application of the slowly varying envelope approximation (SVEA) to the study of the laser with optical feedback, comparing the monochromatic modes obtained using a SVEA approximation with those emerging from a full treatment of the Maxwell-Bloch equations in a coupled cavity formulation of the laser with optical feedback [A. A. Duarte and H. G. Solari Phys. Rev. A **58**, 614 (1998)]. While the SVEA approximation in the body of laser produces reliable results, the same approximation applied to the boundary conditions completely distorts the metamorphosis of the spectrum present in the original model, yet far from the transition region the SVEA gives acceptable results. The failure of the SVEA at the metamorphosis is related to the high sensitivity of the dispersion relations $k(w)$ with respect to frequency changes.

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I. INTRODUCTION

The laser with optical feedback has been an object of study during the last 20 years [1–7] due to its importance for the communication industry as well as the intricacies of its dynamical behavior.

In a recent work [8,9] we have produced an approximation to the laser with optical feedback uniformly valid for values of the reflectivity of the external mirror ranging from 0 to 1. The work unraveled the form in which the monochromatic spectrum of the laser changed from low reflectivity values of the external mirror to large values in a sequence that we have named "the metamorphosis of the monochromatic spectrum."

The metamorphosis of the spectrum is driven by the bifurcations of the solutions associated with the boundary conditions, which in the model have been assumed as a simple vacuum-dielectric interface.

While it is technically possible to produce the discussion of the monochromatic spectrum in terms of Maxwell-Bloch equations, such possibility is greatly reduced for more general solutions. Hence, it is desirable to introduce reasonable simplifications in the equations to allow numerical solutions of the model.

The standard approximations would then be (a) to introduce the slowly varying envelope approximation (SVEA) [10,11] neglecting as usual the differences in frequency of the cavity modes in front of the optical frequency ($\Delta w \ll w$) as well as the differences among wave numbers in front of the wave number of the reference mode ($\Delta k \ll k$), and (b) to discretize the equations.

In the present work we discuss the first approximation leaving the discretization for a future work. The application of the SVEA to Maxwell-Bloch equations in the present case, rather than being a standard step, presents some unexpected aspects that need to be discussed and go beyond the problem of the laser with optical feedback.

We shall show that when applied to the equations in the body of the laser the SVEA introduces negligible errors, but when used in the boundary conditions it gives reliable results only away from the transition region. Furthermore, the process of metamorphosis of the spectrum is completely changed within a full SVEA approximation. However, when the spatial derivatives of the electric field in the active medium are kept in the boundary conditions, the correct results are recovered.

The failure of the SVEA in the transition region is traced to the inability of the approximation to produce bifurcations for the solutions of the equations determined by the boundary conditions emphasizing the need for a careful examination of the boundary conditions associated with partial differential equations.

The rest of the work is organized as follows. Section II introduces the essential details of the model and previous results and is completely based on [8], Sec. III presents the SVEA and the results obtained with two different approximations of the boundary conditions, Sec. IV discusses the reasons behind the failure of the full SVEA and its meaning in terms of the transition matrix, and finally Sec. V presents the concluding remarks.

II. LASER WITH OPTICAL FEEDBACK

We consider a coupled cavity laser. The active medium is confined to the region $0 \leqslant x \leqslant l$ while the region $-L \leqslant x \leqslant 0$ is in the vacuum. A perfect mirror is placed at $x=l$ and a mirror of reflectivity R is placed at $x=-L$.

The dependence of the fields with the transversal coordinates y and z is neglected in what follows, hence the laser is described by the electric field $E(x,t)$, the polarization $P(x,t)$, and the carrier density $N(x,t)$. Note that $P(x,t)=0$ and $N(x,t)=0$ for $-L \leqslant x \leqslant 0$. Furthermore, the carriers in the active medium are assumed to relax to the homogeneous state.

We will summarize here the equations describing the laser with the fields and variables scaled as in [8], we refer for further details to the original references [8,9] and to [12] for the discussion of the susceptibility adopted.

The laser equations in the active medium read

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$$\frac{\partial^2 E(x,t)}{\partial x^2}-\frac{1}{c^2}\frac{\partial^2 E(x,t)}{\partial t^2}=\mu_0\frac{\partial^2 P(x,t)}{\partial t^2}, \tag{2.1}$$

$$\overline{P}(x,\omega)=\epsilon_0\chi(\omega,N)\overline{E}(x,\omega), \tag{2.2}$$

$$\frac{dN(t)}{dt}=-\gamma_\parallel N(t)+J+\frac{2i}{\hbar l}\int_0^l dx[E(x,t)P^*(x,t) -E^*(x,t)P(x,t)], \tag{2.3}$$

where $\overline{P}(x,\omega)$ and $\overline{E}(x,\omega)$ indicate the Fourier transform with respect to the time of the polarization and electric field. The expression of $\chi(\omega,N)$ reads [8,9,12].

$$\chi(\omega,N)=C_0\left[-\frac{N}{1-i\Omega}+C_1(\sqrt{\Omega+i}-iC_2)\right] \tag{2.4}$$

where $\Omega=(\omega-\omega_g)/\gamma_\perp$ is the dimensionless detuning, $C_0=i\Gamma g^2/\epsilon_0\hbar\gamma_\perp$, $C_1=(2m\gamma_\perp/\hbar)^{3/2}/2\pi N_T$, and $C_2=2(2\hbar/\gamma_\perp ma^2)^{1/2}$. Here ω_g is the frequency associated with the energy gap in the electronic bands of the semiconductor, a is the lattice parameter of the semiconductor medium, g is the electric dipole element of the active media, m is the effective mass of the electron, Γ is the confinement factor, and $\gamma_\perp$ is the decay rate associated with the polarization, and $\gamma_\parallel$ is the decay rate of the carrier density.

The electromagnetic field in $-L\leqslant x\leqslant 0$ is described by

$$E(x,t)=-RG(t-\tau-x/c)+G(t+x/c) \tag{2.5}$$

with $\tau=2L/c$ and G being and arbitrary C^2 function.

The description is completed with the boundary conditions,

$$E(l,t)=0, \tag{2.6}$$

$$\lim_{\epsilon\to 0}E(-\epsilon,t)=\lim_{\epsilon\to 0}E(\epsilon,t), \tag{2.7}$$

$$\lim_{\epsilon\to 0}\partial E(x,t)/\partial x|_{x=-\epsilon}=\lim_{\epsilon\to 0}\partial E(x,t)/\partial x|_{x=\epsilon}. \tag{2.8}$$

Given that the susceptibility is space independent, as N is, the monochromatic solutions of the wave equation (2.6)–(2.8) can always be written in the form

$$E(x,t)=\exp(-i\omega t)[E_+\exp(ik_1x)+E_-\exp(-ik_1x)] \tag{2.9}$$

and, correspondingly, for the electric field in the vacuum we use $G(t+x/c)=H_0\exp[i\omega(t+x/c)]$, where ω is the field frequency, and the complex wave number k_1 verifies the dispersion relation

$$k_1=k_0\sqrt{1+\chi(\omega,N)}, \tag{2.10}$$

$k_0=\omega/c$ being the field wave number in the vacuum. On the other hand the boundary conditions (2.6)–(2.8) imply the following condition for k_1:

$$\mathcal{C}(k_0,k_1,R,L,l)\equiv k_0[\{\exp(2ik_1l)-1\}\{R\exp(2ik_0L)+1\}] +k_1[\{\exp(2ik_1l)+1\}(R\exp(2ik_0L)-1\}]=0 \tag{2.11}$$

and the relation

$$E_-=-\exp(2ik_1l)E_+. \tag{2.12}$$

Equations (2.10) and (2.11) allow to solve for (k_1,N) as functions of (ω,R,L,l). These solutions yield a complex N in general and, hence, the physical solutions are those given by the implicit relation $\mathrm{Im}(N)=0$. By substituting this value of N into Eq. (2.3), and making use of Eq. (2.12), the value of $|E_+|^2$ is obtained as a function of (N,J). In particular, the laser threshold for each mode is determined by setting $|E_+|^2=0$ and reads $J_{threshold}=\gamma_\parallel N$.

In [8] we introduced two complementary iterative algorithms for solving Eq. (2.11). We shall name the stable fixed points of the algorithm

$$k_1^{m+1}(q)=\frac{1}{2li}\ln\left[-\frac{-k_0(W+1)+k_1^m(W-1)}{k_0(W+1)+k_1^m(q)(W-1)}\right]+\frac{q\pi}{l} \tag{2.13}$$

[where $W=R\exp(i2k_0L)$] the *ordinary set* of solutions, while the solutions that are obtained iterating the inverse relation

$$k_1^{m-1}=-k_0\frac{[\exp(i2k_1^ml)-1](W+1)}{[\exp(i2k_1^ml)+1](W-1)} \tag{2.14}$$

are named the *extraordinary* set.

It has been previously shown that for $R\sim 0$ and $R\sim 1$ only the ordinary set is relevant, however, the extraordinary set plays a central role in the metamorphosis of the spectrum.

We shall now turn to the discussion of the monochromatic solutions as they arise in the SVEA.

III. SLOWLY VARYING ENVELOPE APPROXIMATION

The slowly varying envelope approximation is a well-known approximation in laser physics [10,11] and we refer to the laser physics texts for the details.

It suffices here to say that the electric field and polarization associated with the modes that fall under the gain curve can be approximated in the active medium by

$$E=\exp(-i\omega_0t)[\exp(iKx)E_+(x,t)+\exp(-iKx)E_-(x,t)], \tag{3.1}$$

$$P=\exp(-i\omega_0t)[\exp(iKx)P_+(x,t)+\exp(-iKx)P_-(x,t)], \tag{3.2}$$

where $|\partial E_+(x,t)/\partial x|\ll|E_+||K|$, $|\partial E_-(x,t)/\partial x|\ll|E_-||K|$, $|\partial E_+(x,t)/\partial t|\ll|E_+||\omega_0|$, and $|\partial E_-(x,t)/\partial t|\ll|E_-||\omega_0|$ and similar expressions for $P_\pm$. We will take for definiteness the values of K and ω_0 as those corresponding to the stable solution of the Maxwell-Bloch equations coupled with Eqs. (2.11) for $R=0$. Note that the electric field in the vacuum can correspondingly be written using $G(t+x/c)=H(t+x/c)\exp[i\omega_0(t+x/c)]$.

We can now turn to examine the corresponding approximation for the boundary conditions (2.7) and (2.8). For $x=0$ we have

$$E_+(0,t)+E_-(0,t)=H(t)-Re^{i\omega_0\tau}H(t-\tau), \tag{3.3}$$

$$\begin{aligned} iK(E_+(0,t)-E_-(0,t))+\partial_x E_+(0,t)+\partial_x E_-(0,t) \\ =\frac{1}{c}\partial_t H(t)-i\frac{\omega_0}{c}H(t)+\frac{R}{c}e^{i\omega_0\tau} \\ \times(\partial_t H(t-\tau)-i\omega_0 H(t-\tau)), \end{aligned} \tag{3.4}$$

and for $x=l$,

$$E_+(l,t)e^{iKl}=-E_-(l,t)e^{-iKl}. \tag{3.5}$$

In principle we can disregard the spatial and temporal derivatives in front of the other terms in Eq. (3.4) obtaining

$$\begin{aligned} iK[E_+(0,t)-E_-(0,t)] \\ =-i\frac{\omega_0}{c}[H(t)+Re^{i\omega_0\tau}H(t-\tau)]. \end{aligned} \tag{3.6}$$

To test the validity of the latter approximation we shall compare the results obtained using Eq. (3.6) with those obtained using the alternative equation

$$\begin{aligned} iK[E_+(0,t)-E_-(0,t)]+\partial_x E_+(0,t)+\partial_x E_-(0,t) \\ =-i\frac{\omega_0}{c}[H(t)+Re^{i\omega_0\tau}H(t-\tau)]. \end{aligned} \tag{3.7}$$

where the spatial derivatives of the envelopes in the active medium have been kept.

It is important to notice at this point that when Eq. (3.6) is applied to a monochromatic solution, Eq. (2.11) is replaced by

$$\begin{aligned} \frac{\omega_0}{c}\{[\exp(2ik_1 l)-1][R\exp(2ik_0 L)+1]\} \\ +K\{[\exp(2ik_1 l)+1][R\exp(2ik_0 L)-1]\}=0, \end{aligned} \tag{3.8}$$

which determines k_1 as an explicit function of ω

$$k_1(q)=\frac{1}{2li}\ln\left(-\frac{-\omega_0(W+1)+cK(W-1)}{\omega_0(W+1)+cK(W-1)}\right)+\frac{q\pi}{l} \tag{3.9}$$

Equation (3.9) is an approximation for the ordinary set of solutions (2.13) but not for the extraordinary set (2.14), a fact that anticipates that the behavior of this approximation will not be satisfactory in the critical region of the metamorphosis and further justifies the name "ordinary" given to the solutions of Eq. (2.11) obtained with Eq. (2.13).

In contrast with the approximation (3.6), keeping the spatial derivatives of the field in the active medium as in Eq. (3.7), we keep both sets (ordinary and extraordinary) of solutions.

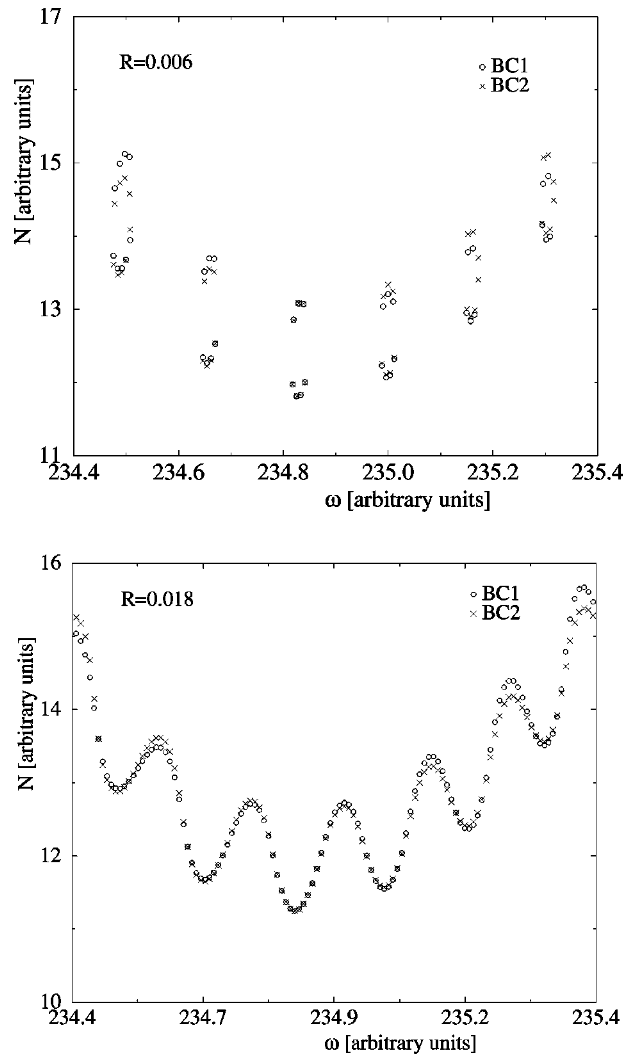

FIG. 1. Upper panel: Monochromatic spectrum below the metamorphosis obtained using Eqs. (3.7) (BC1, circles) and (3.6) (BC2, crosses). Lower panel: The spectrum above the transition region using Eqs. (3.7) (BC1, circles) and (3.6) (BC2, crosses). The parameter values adopted in numerical calculations are $\omega_g=235\gamma_\perp$, $g/e=0.2$ Å, $a=5$ Å, $\gamma_\perp=10^{13}$ Hz, $\gamma_\parallel=10^9$ Hz, $\Gamma=0.4$, $C_1=11.4$, $C_2=9.6$, $l=10c/\gamma_\perp\sim300$ μm, and $L=35l$.

The spectra obtained with both approximations of the boundary conditions are compared in Figs. (1) and (2).

Away from the region of the metamorphosis, the results obtained using Eqs. (3.7) and (3.6) do not differ significantly (see Fig. 1) and compare well with the corresponding results obtained without approximations in [8].

In contrast, we can realize from the Fig. 2 that when the approximated boundary conditions (3.6) are used, the description of the monochromatic spectrum in the transition region is completely (qualitatively) different from the spectrum obtained without approximations, while the approximation (3.7) does not introduce important errors. Moreover, the spectrum obtained with Eq. (3.7) is qualitatively and quantitatively similar to the one obtained for the same values of R without the SVEA approximation in [8].

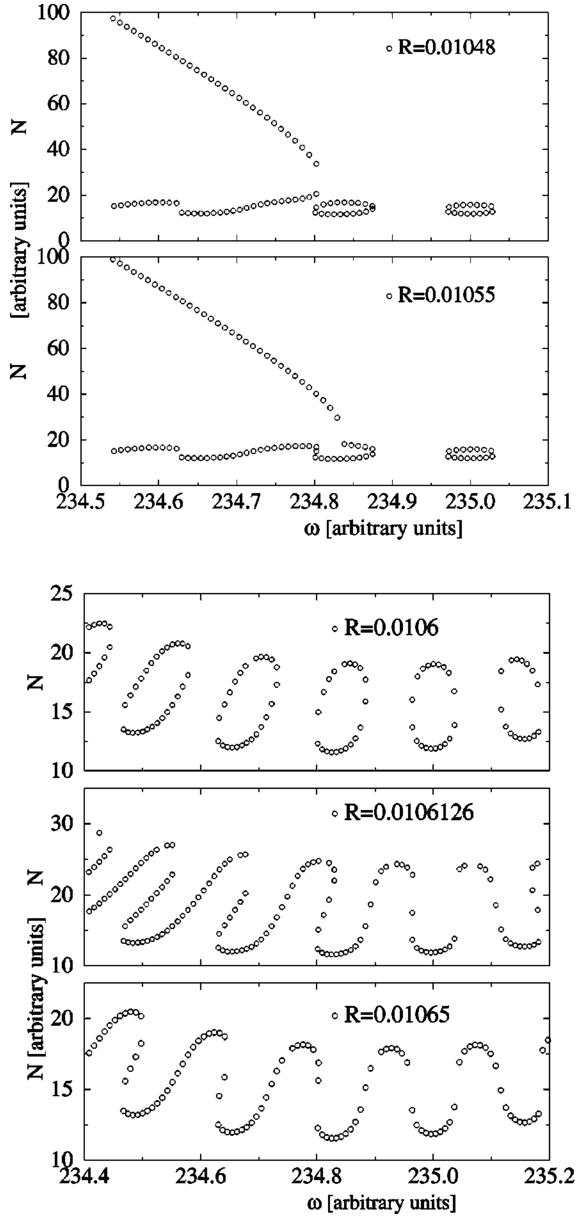

FIG. 2. The metamorphosis of the monochromatic spectrum. Above: Results obtained using Eq. (3.7). The upper and lower panels correspond with the spectrum obtained for R just below ($R=0.010\,48$) or just above ($R=0.010\,55$) the transition. Below: Results obtained using Eq. (3.6). Note that the metamorphosis takes place for different values of the reflectivities ($0.010\,60\leqslant R \leqslant 0.010\,65$) and, most importantly, in a completely different way.

IV. THE SVEA IN THE TRANSITION REGION OF THE SPECTRUM

A. Discussion of results

The results presented in Sec. III make it clear that it is necessary to keep the contribution of the spatial derivatives of the electromagnetic field in the active medium to reproduce the exact results in the critical region, while the standard approximation (3.6) is correct for parameter values that are away from the transition region.

We can further track the reasons for the failure in the critical region. The approximation (3.6) produces very poor results for those modes that belong to the extraordinary set of solutions since it is equivalent to apply the iterative procedure (2.13) once using K as the initial value. For those wave vectors of the extraordinary set, such procedure will deteriorate rather than improving the estimate.

As observed in [8] the solutions of the extraordinary set present a strong dependence of the wave vector with respect to the frequency in the transition region, it is this dependence that cannot be approximated with the standard procedure.

Away from the critical region, the population inversion required for the extraordinary set of solutions to be realized is very large and we can, in principle, disregard them. In particular, for $R\rightarrow 1$ the extraordinary set has modes with associated carrier densities that grow to infinity and can be completely neglected because of this reason.

B. The transmission matrix

The interface between two different media can also be described in terms of reflection r and transmission coefficients t. These coefficients describe the effect of the interface on the different wave amplitudes arriving or departing from it. Then, we define a set of coefficients (r,t) for each side of the interface. We shall refer to these coefficients as (r,t) or (r',t') depending on whether the set of coefficients refer to the left or to the right of the interface. In this case,

$$E_+ = r'E_- + tH_+ , \tag{4.1}$$

$$H_- = t'E_- + rH_+ , \tag{4.2}$$

where E_+ (E_-) is the amplitude of a right (left) traveling waves inside the semiconductor while H_+ (H_-) is the amplitude of a right (left) traveling wave in the vacuum.

This description can be contrasted against that pursued in terms of the boundary conditions. Consider a monochromatic mode of frequency ω and wave vector k inside the cavity. Recall that in our analysis the wave number outside the semiconductor is ω/c. Then, from Eqs. (2.7) and (2.8) we arrive at

$$E_+ + E_- = H_+ + H_- , \tag{4.3}$$

$$k(E_+ - E_-) = \frac{\omega}{c}(H_+ - H_-). \tag{4.4}$$

After relating both descriptions,

$$r = \frac{\omega - kc}{\omega + kc}, \tag{4.5}$$

$$r' = \frac{kc - \omega}{\omega + kc}, \tag{4.6}$$

$$t = \frac{2\omega}{\omega + kc}, \tag{4.7}$$

$$t' = \frac{2kc}{\omega + kc}. \tag{4.8}$$

Note that ω and k refer to monochromatic mode.

If instead of the boundary conditions (2.8) we had used the approximated boundary conditions (3.6) disregarding the partial derivatives of electromagnetic field, we would have arrived to the same expressions but with $k=K$ and $\omega=\omega_0$, hence, the transmission and reflection coefficients would be approximated by constants.

We know from the previous section that this is the case for small R (below the metamorphosis). However, during the metamorphosis this approximation is not possible. If instead of Eq. (3.6) we use the approximate expression for the boundary conditions keeping the spatial derivative of the electromagnetic field inside the active medium, the coefficients given by Eqs. (4.5)–(4.8) depend on k, i.e., a constant transition-matrix approximation is no longer possible.

Hence, a model based on a constant transition-matrix description will fail to capture the metamorphosis suffered by the spectrum of monochromatic modes of this system.

V. CONCLUDING REMARKS

We have studied with some detail the slowly varying envelope approximation in a laser with optical feedback described in terms of Maxwell-Bloch equations in a coupled cavity.

Thanks to the knowledge acquired in [8] of the metamorphosis of the spectrum we have found, not without surprise, that the correct approximation of the boundary conditions in the interface requires to keep the contribution of the partial derivatives of the electromagnetic fields in the active medium.

This requirement relates to the proper description of the ‘‘extraordinary’’ set of solutions that plays a relevant role in the metamorphosis of the spectrum but can be neglected away from the critical region.

Furthermore, since the approximation of the boundary conditions in terms of a transition matrix presents exactly the same problems as our approximation (3.6), which indeed turns to be a constant transition-matrix approximation, we conclude that such procedure will also distort completely the metamorphosis of the spectrum.

Taking a more general point of view, it might be discussed whether a simple dielectric-vacuum interface is a realistic description of the interface between the active region of a (semiconductor) laser and the vacuum or not. However, the moral of this work is that a careful and realistic description of the laser-vacuum interface is required if the behavior of the laser in the critical (metamorphosis) region of the laser behavior is going to be achieved.

Traditionally, boundary conditions have not received much attention when considering partial differential equations and to some extent, the present work shows that for most parameter values they do not play a central role. However, we have found that our standard ‘‘*a priori*’’ estimates for the approximation of the boundary equation were not always correct.

We have also shown that the interface between the laser and the vacuum is deeply involved in the description of the laser with feedback in the region of the metamorphosis of the spectrum.

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