Metamorphosis of the monochromatic spectrum in a double-cavity laser as a function of the feedback rate

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We discuss the changes in the spectrum of monochromatic modes in a double-cavity laser showing how the characteristic frequencies change from those proper of a short cavity into those proper of a long cavity using as control parameter the reflectivity of an ''external'' mirror. The problem is cast into the language of bifurcations in a nonlinear eigenvalue problem. The results show that the transition is mostly dictated by the boundary conditions and occurs regardless of the laser model. Hence, the study presented can be extended to other optical cavities and boundary value problems. Limits of validity of simpler double-cavity laser models, such as the Lang-Kobayashi, are drawn. [S1050-2947(98)10706-0]

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I. INTRODUCTION

The bifurcations through which complexity arises in extended systems are one of the central subjects of present studies in nonlinear dynamics. In this context, the dependence of the solutions with respect to the boundary conditions deserves special attention. In this work we consider the changes affecting the solutions of a laser with optical reinjection as a function of the boundary conditions. The choice of system has to do, in part, with its relevance in technological applications.

When a semiconductor laser forms part of a complex communication device there is a reasonable possibility for the light emerging from the laser to be reflected in some other element of the device thus reentering the laser. It is known that even small amounts of reentering reflected light can destabilize the laser, hence producing temporal oscillations of various types such as ''low frequency fluctuations'' [2,3] and ''coherent collapse'' [3–6].

The academic version of this problem consists in the study of a laser (semiconductor if possible) with an external [7] mirror, conforming in such a form a ''double-cavity laser'' [1–6,8]. The position of the external mirror and its reflectivity are considered parameters of the problem.

Let us consider a laser of length l ($l \sim 10-300\ \mu$m) with an external mirror of reflectivity R located at a distance L ($L \gg l$). There are two clear limit cases, when $R=0$ the distance between the monochromatic modes is $\Delta\omega \sim v/l$, with v the speed of the light in the semiconductor dielectric; when $R=1$ the distance between laser modes is $\Delta\omega \sim c/L$ with c the speed of light in the vacuum.

For reflectivity values close to the limit cases we can attempt perturbative solutions [9]. For $R\sim 1$ the laser losses are expected to be very small and the solutions can be obtained as expansions in the empty cavity modes [8]. For $R=0$ the laser works at a single (stable) longitudinal mode, hence, for $R\sim 0$ it can be assumed to operate near (in terms of frequencies) this mode and it is plausible to neglect all other monochromatic modes of the semiconductor laser.

The $R\sim 0$ case has been considered frequently in the literature [1–6] in terms of time-delayed ordinary differential equations. This is under the assumption that (1) the spatial extension of the semiconductor can be neglected assuming a ''single'' (longitudinal) mode operation, and (2) that the reflection in the external mirror is represented by a term proportional to the electric field delayed at time $\tau=2L/c$. The Lang-Kobayashi [1] equations are the paradigm of this presentation.

We will avoid these simplifying hypotheses and we will describe the semiconductor medium in its full spatial extension, hence allowing for ''longitudinal multimode'' operation. We will also consider the feedback a consequence of the boundary conditions in the external mirror and in the semiconductor-vacuum transition (thus avoiding the second simplifying hypothesis). Without the second assumption the restriction of the problem to the semiconductor media results in a set of partial differential equations (PDE's) with time-delayed boundary conditions. Having lifted two hypotheses we are in conditions to check when they are satisfied and when they are no longer reasonable hypotheses. Hence, we can draw a limit of validity for the Lang-Kobayashi equations.

Some of our motivating questions for this study are the following: How does the spectrum change (the metamorphosis) when R is increased from 0 to 1? Do boundary conditions rule completely the bifurcations or, contrarily, is the laser physics determinant? (We shall see that the laser plays a secondary role and the boundary conditions rule the metamorphosis.) What kind of bifurcations (if any) enter in the process? All these questions, and others, will find answers in what follows.

The rest of the work is organized as follows. Section II formulates the problem as a nonlinear eigenvalue problem in laser physics. Section III discusses how the monochromatic modes are found. Although this section is basically technical, we introduce here the language of multivalued functions that will be used in the discussion of the problem and discuss the solutions of the equations emerging from the boundary conditions. Section IV describes the sequence of bifurcations of monochromatic solutions that occur when the reflectivity is increased from $R=0$ up to $R=1$. Section V presents the concluding remarks.

II. LASER WITH OPTICAL FEEDBACK

Let us begin by presenting the laser equations in a double cavity. We consider a laser that extends in the x direction; the active medium is located at $0 \leqslant x \leqslant l$ while the external mirror of reflectivity R is placed at $x=-L$. The dependence of the fields with the transversal coordinates, (y,z), is neglected in what follows, hence, the laser is described by the electric field $E(x,t)$, the polarization $P(x,t)$ $[P(x,t)=0$ for $-L \leqslant x < 0]$, and the carrier density $N(x,t)$ $[N(x,t)=0$ for $-L \leqslant x < 0]$. The electric field satisfies Maxwell's equations

$$\frac{\partial^2 E(x,t)}{\partial x^2} - \frac{1}{c^2}\frac{\partial^2 E(x,t)}{\partial t^2} = \mu_0 \frac{\partial^2 P(x,t)}{\partial t^2}, \tag{1}$$

while the material field satisfies

$$\frac{\partial N(x,t)}{\partial t} = -\gamma_{\parallel} N(x,t) + J + D\Delta N(x,t) + \frac{2i}{\hbar}[E(x,t)P^*(x,t) - E^*(x,t)P(x,t)]. \tag{2}$$

The different variables and operators have the following meaning: J=current pumping the carriers, D=diffusion coefficient for carriers, ΔN=Laplacian of N, $\gamma_{\parallel}^{-1}$ =nonradiative decay time for the carrier density ($\sim 10^9$ Hz).

The boundary conditions complete our set of equations; they are (i) the electric field vanishes at the (perfect) mirror

$$E(l,t)=0; \tag{3}$$

the carriers cannot leave the semiconductor

$$\partial N(x,t)/\partial x|_{x=l} = \partial N(x,t)/\partial x|_{x=0}; \tag{4}$$

the electric and the magnetic fields are continuous (assuming that the semiconductor presents no magnetic polarization)

$$\lim_{\epsilon \to 0} E(-\epsilon,t) = \lim_{\epsilon \to 0} E(\epsilon,t), \tag{5}$$

$$\lim_{\epsilon \to 0} \partial E(x,t)/\partial x|_{x=-\epsilon} = \lim_{\epsilon \to 0} \partial E(x,t)/\partial x|_{x=\epsilon}. \tag{6}$$

The reflectivity of the external mirror is R. This condition has to be written in terms of the general solution of Maxwell's equation in the vacuum, $E(x,t)=A_+(ct-x)+A_-(ct+x)$, for $x \leqslant 0$, where A is an arbitrary C^2 function and c the speed of light. The condition reads

$$A_+(ct+L) = -RA_-(ct-L) \tag{7}$$

or $E(x,t)=-RA[c(t-2L/c)-x]+A(ct+x)$. Note that having written the general solution for the electric field in $x \in [-L,0]$ we need to seek solutions to Eq. (1) only for $x \in [0,l]$. In this setting the problem becomes a PDE with boundary conditions involving time delays.

In what follows we will further restrict our study to the case in which the diffusion of carriers is very fast, i.e., D large in Eq. (2). Neglecting the spatial dependence of the carrier density $N(x,t)=N(t)$ in Eq. (2) we obtain

$$\frac{dN(t)}{dt} = -\gamma_{\parallel} N(t) + J + \frac{2i}{\hbar l}\int_0^l dx (E(x,t)P^*(x,t) - E^*(x,t)P(x,t)). \tag{8}$$

This approximation is compatible with the boundary conditions.

The relation between E, P, and N completes the equation set defining the characteristics of the active material. Under the assumption of a quadratic energy dependence (with respect to the electron wave vector) the following relation has been introduced in [10,11] for the dielectric susceptibility, $P(x,\omega)=\epsilon_0\chi(\omega,N)E(x,\omega)$, as a function of the frequency ω and the carrier density $N(\omega)$:

$$\chi(\omega) = \frac{\Gamma|g|^2}{\epsilon_0 \hbar}\left\{ \frac{1}{\pi^2}\left(\frac{2m}{\hbar}\right)^{3/2} \times \left(i\frac{\pi}{2}\sqrt{z(\omega)} + \sqrt{\frac{\hbar k_m^2}{2m}}\right) + \frac{N(\omega)}{z(\omega)}\right\}, \tag{9}$$

with $z(\omega)=\omega-\omega_g+i/T$, T the characteristic decay time of the polarization, and ω_g the frequency associated with the energy gap in the electronic bands of the semiconductor. The values adopted in numerical calculations are $g/e=0.20$ Å, $\omega_g=2.35\times 10^{-15}$ Hz, $\Gamma=0.4$, $1/T=10^{13}$ Hz, $k_m=\pi/5$ Å^{-1}, $l=300$ μm, and $L=35l$.

Monochromatic solutions for the laser in a double cavity can be found by proposing

$$E_j(x,t) = \exp(-i\omega t) \times [A_j \exp(ik_j x) + B_j \exp(-ik_j x)], \tag{10}$$

where the subscript j indicates the region: $j=0$ for the vacuum and $j=1$ for the active media.

Since the boundary conditions (3,5,6,7) are linear in E, the nontrivial, $E \neq 0$, monochromatic solutions satisfy

$$\begin{aligned}\mathcal{C}_B(k_0,k_1,R,L,l) &\equiv k_0\{[\exp(2ik_1 l)-1][R\exp(2ik_0 L)+1]\} \\ &\quad + k_1\{[\exp(2ik_1 l)+1][R\exp(2ik_0 L)-1]\}=0,\end{aligned} \tag{11}$$

we will refer to Eq. (11) as the "boundary equation" for simplicity.

The Maxwell equations (1) are also linear in E, since we have assumed a linear susceptibility; for $E \neq 0$ they are satisfied if and only if

$$k_0^2 = \omega^2/c^2, \tag{12}$$

$$-k_1^2 + \frac{\omega^2}{c^2} = -\frac{\omega^2}{c^2}\chi(\omega,N). \tag{13}$$

III. BIFURCATIONS ASSOCIATED WITH THE BOUNDARY CONDITIONS

The only equation where E enters in a nonlinear way is Eq. (8) and it will determine the amplitude, A_j, B_j in Eq. (10), of the electric field as a function of the pumping current J. The current associated with a zero amplitude defines the current at threshold for the mode considered. Equation (8) does not convey information about the spectrum of monochromatic modes in the laser. We will not turn to this equation for the rest of the discussion; the only reference to the material media left is encrypted in the dielectric constant.

Hence, the remaining task is to solve a set of five real equations [Eq. (12) is real while Eqs. (13) and (11) are complex equations] for five real variables (ω, k_0, N, which are real, and k_1, which is complex), hence, the equation will have solutions, in general, only at isolated values: the values corresponding to the monochromatic modes.

We like to think of Eqs. (11)–(13) in the following form: Eq. (12) defines the function $k_0(\omega)=\omega/c$, Eq. (11) defines the function $k_1(k_0)=k_1(\omega/c)$ (which is multivalued, i.e., has different branches), and Eq. (13) defines the function $N(\omega,k_0,k_1)=N(\omega)$; $N(\omega)$ evaluates to complex numbers in general. Since the carrier density is a real number the condition $\mathrm{Im}[N(\omega)]=0$ must be imposed. This latter condition cannot be satisfied for general values of ω and determines the frequencies of the monochromatic solutions.

We look for solutions of Eq. (11) using a recursive algorithm,

$$k_1^{m+1}(q)=\frac{1}{2li}\ln\left(-\frac{-k_0(W+1)+k_1^m(W-1)}{k_0(W+1)+k_1^m(q)(W-1)}\right)+\frac{q\pi}{l}, \tag{14}$$

where $W=R\,\exp(i2k_0L)$ and q labels the branch of the solution; i.e., for fixed values of the parameters and the frequency $w=k_0c$ there is a numerable set of solutions to the boundary conditions (11) labeled by q.

The fixed points of the map (14) are the solutions of Eq. (11). The iterative procedure converges for most values of the parameters and the frequency. Since Eq. (14) defines a one-dimensional complex map, either Eq. (14) converges or its inverse map

$$k_1^{m-1}=-k_0\frac{[\exp(i2k_1^m l)-1](W+1)}{[\exp(i2k_1^m l)+1](W-1)} \tag{15}$$

converges or the derivative of the map at the fixed point has an absolute value of 1.

It is an interesting fact that only in the region of the metamorphosis of the spectrum is the inverse algorithm (15) needed.

The different branches of solutions of Eq. (11) are clearly separated whenever the conditions for the implicit function theorem are satisfied; in this case we need

$$\frac{\partial \mathcal{C}_B(k_0,k_1,R,L,l)}{\partial k_1}\neq 0 \tag{16}$$

at the fixed point of Eq. (14).

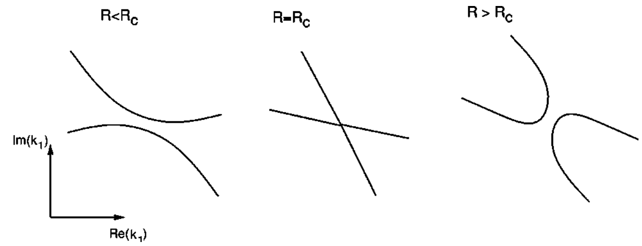

FIG. 1. Bifurcation of the solutions of the boundary equations at (k_1^c,k_0^c,R^c). The curves represent the trace of k_1 as a function of ω for three different values of the reflectivity R.

When the hypotheseis of the implicit function theorem (16) are not satisfied, say at $k_1=k_1^c$, $k_0=k_0^c$, $R=R^c$, the expression resulting from the boundary conditions (11) reads

$$\mathcal{C}_B(k_0,k_1,R,L,l)=a(k_0,R,L,l)+b(k_0,R,L,l)(k_1-k_c)^2+O(|k_1-k_c|^3) \tag{17}$$

with $a(k_0^c,R^c,L,l)=0$ and $b(k_0^c,R^c,L,l)\neq 0$. Note that there are two real conditions to be satisfied at the critical point and this requires us to adjust two parameters. We have arbitrarily chosen k_0 and R [the bifurcation set is a one-dimensional object in the three-dimensional space (ω,W)]. The change produced in the solutions of Eq. (11) by the bifurcation at the critical point (k_1^c,k_0^c,R^c) is schematically presented in Fig. 1.

Equation (17) represents the local change of the solutions of the boundary equation (11). This description has to be integrated in the global picture that displays several branches of solutions.

Instead of discussing the changes as a function of one parameter it is useful to notice that R and L appear only in Eq. (11) and always in the form $W\equiv R\,\exp(2i\omega L)$; hence we shall consider the dependency of solutions with respect to the complex parameter W (this situation can also be thought as the $L\to\infty$ limit).

For every value of the frequency ω, Eq. (11) defines an implicit map in the complex plane that relates the image k_1 to its preimage W. Since $|W|=R$ defines a circle, for fixed R and ω the boundary equation defines the images of this circle. The locus of this image is given by the level curves

$$R=|W|=\left|\frac{k_0(z-1)-k_1(z+1)}{k_0(z-1)+k_1(z+1)}\right|. \tag{18}$$

The levels curves are presented in Fig. 2 for different values of R.

Note that for $R=0$ the ''circle'' $|W|=0$ is actually reduced to a point; correspondingly, its images are points. For $R\neq 0$ the structure of the circle becomes visible and the images of the circle are closed curves.

Notice further that for different values of R the images of the circle revolve around one or more image points of $W=0$ (the number of points enclosed depends on the value of R).

The change in the number of points enclosed by a given image of the circle occurs at some critical values of R where two different images of the circle touch each other and merge according to the local mechanism described by (17) [12].

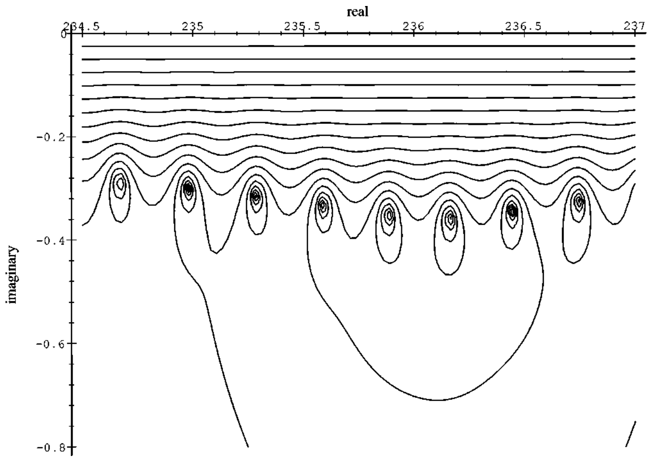

FIG. 2. Complex wave vector k_1 allowed for different values of R. Contour lines are labeled by R and increase from $R=0$ (points) to $R=1$ (real axis).

IV. METAMORPHOSIS OF THE SPECTRUM OF MONOCHROMATIC SOLUTIONS

Having examined the bifurcations of solutions of the equations associated with the boundary conditions we turn to the problem of how these bifurcations and others alter the spectrum of monochromatic solutions. The discussion is organized for R increasing from 0 to 1.

We recall that, after solving the equations associated with the boundary conditions the frequencies of the monochromatic modes are those determined by the implicit equation $\mathrm{Im}[N(\omega)]=0$ (see Sec. III for definitions) since the carrier density N is a real number.

For $R=0$ the graph of $\mathrm{Im}[N(\omega)]$ consists of a set of lines with negative slope. The lines correspond to different solutions of the boundary conditions and can be labeled, for fixed ω, by an integer q as in Eq. (14). The intersections of these lines with the ω axis determine the frequencies of the monochromatic modes; see Fig. 3. The mode with minimal value of $\mathrm{Re}[N(\omega)]$ is the stable (lasing) solution of the laser without optical reinjection (see [11] for a discussion). Note also that the Lang-Kobayashi equations [1] for $R=0$ present only this stable solution while all other modes have been neglected.

For $R>0$ the $\mathrm{Im}[N(\omega)]$ lines acquire a modulation that increases in depth with increasing R. The period of the modulation is determined by the ratio L/l. In Fig. 4 a few more monochromatic solutions have emerged through saddle-node bifurcations. These new solutions are clustered around the modes of the laser without optical reinjection in the $\mathrm{Re}(N)(\omega)$ plot (see Fig. 4) and we used to refer to them as ''islands.'' They are the finite L version of the circles emerging from the points in our discussion of the bifurcations of solutions of the boundary equations (Sec. III). The island emerging from the stable solution of the laser without optical feedback approximately correspond to the solution of the Lang-Kobayashi equation (see Fig. 6 of [2] and Fig. 1 of [3]).

Increasing the reflectivity to $R=0.010\,48$ the plot $N(\omega)$ changes significantly, see Fig. 5. We can see in Fig. 5 that for low frequencies the islands have merged into one, while for high frequencies the structure of the modes still resembles a set of islands. This change is the direct consequence of the bifurcations of solutions of the boundary equations discussed in Sec. III. For small R each island was linked to a single branch of the solutions of Eq. (11); the bifurcations changing the multivalued solutions of Eq. (11) reflect similar changes in the spectrum of monochromatic modes, i.e., they force the merging of islands.

The graph of $\mathrm{Im}[N(\omega)]$ is also telling; see Fig. 6. The lines corresponding to different branches of solutions are no longer roughly parallel. There are some lines that intersect several branches of solutions; these lines are associated with solutions of the algorithm (15) while their change of slope (from negative to positive or vice versa) is roughly related to marginally stable solutions of the maps (14) and (15).

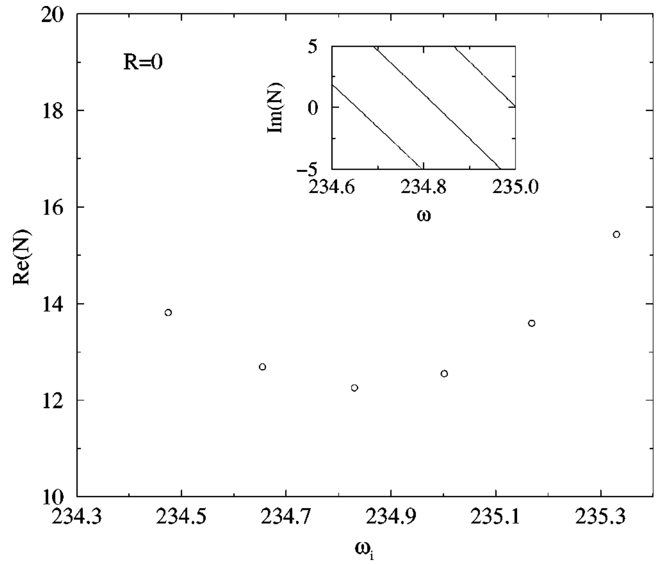

FIG. 3. Carrier density vs frequency for the laser without external mirror. Inset: the frequencies of the monochromatic modes correspond to the intersection of the lines $\mathrm{Im}[N(\omega)]$ with the ω axis. The frequencies are measured in units of $1/T$.

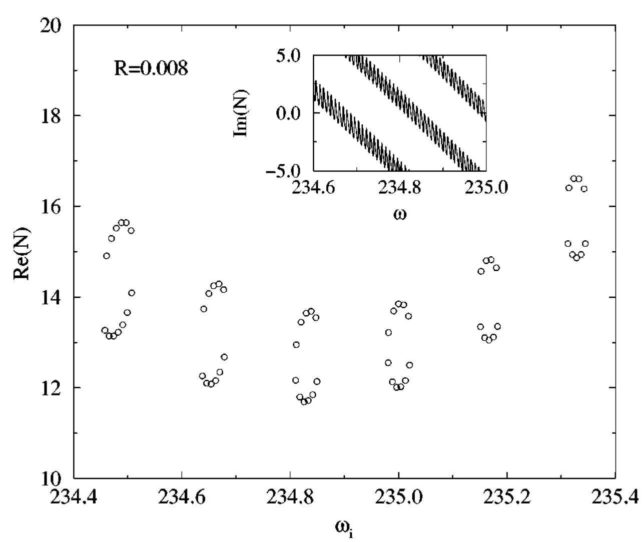

FIG. 4. Islands of solutions in the carrier density vs modal frequency plot for $R=0.008$. Inset: the frequencies of the monochromatic modes correspond to the intersection of the lines $\mathrm{Im}[N(\omega)]$ with the ω axis.

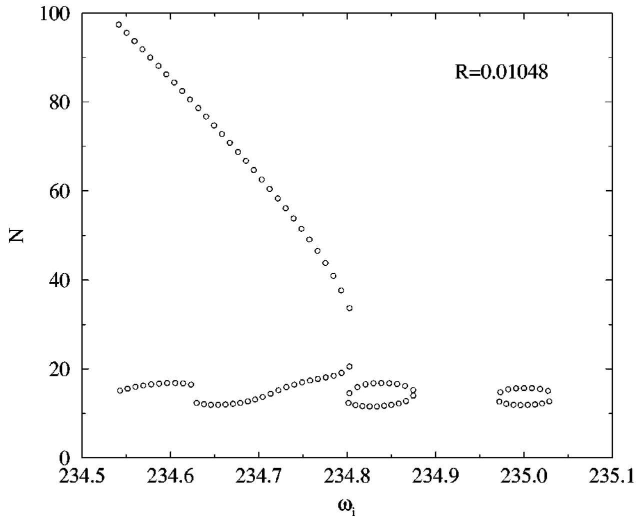

FIG. 5. Carrier density as a function of the frequency for a transitional case ($R=0.010\,48$, $l=300$ μm, $L=35\times l$).

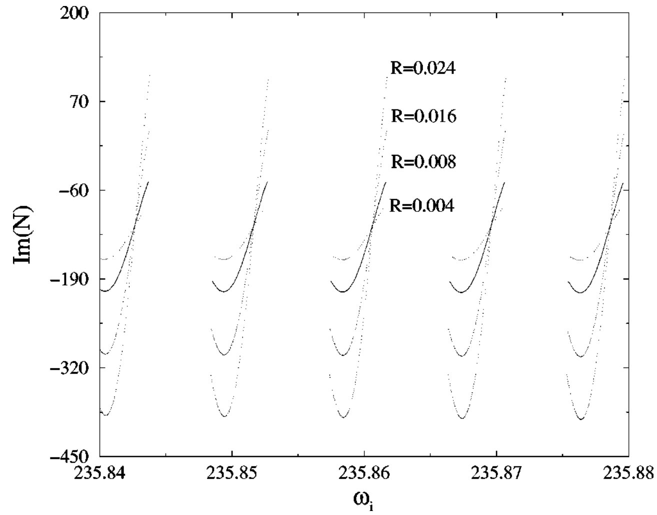

FIG. 7. Creation of new modes through saddle-node bifurcations increasing the value of R.

The solutions obtained with the algorithm (15) are displayed in Fig. 7 for several values of R. Since the appearance of new monochromatic modes (or their disappearance) corresponds to the appearance or disappearance of intersections of Im$[N(\omega)]$ with the ω axis, we can see in Fig. 7 how these changes are operated through successive saddle-node bifurcations occurring in a very small region of parameter space.

It is important to realize that the description of these changes in the spectrum of monochromatic modes is beyond the possibilities of the Lang-Kobayashi equations since it requires the interaction in bifurcations of solutions coming from different islands, i.e., from different modes of the laser without feedback, and there is no room for them in the Lang-Kobayashi equations.

For values of R beyond the transition region, $R>0.011$ the spectrum of monochromatic modes is represented by a single wavy line. Increasing the value of R the waviness disappears gradually and the spectrum takes a form that closely resembles the situation for $R=1$; see Fig. 8.

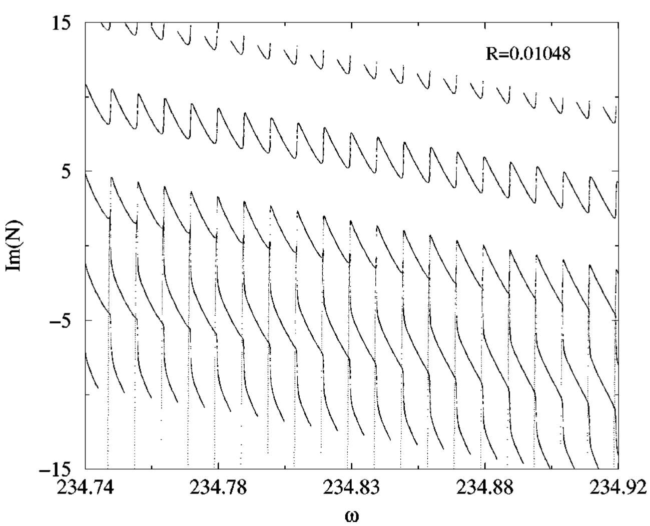

FIG. 6. Function Im$[N(\omega)]$ for $R=0.010\,48$. Different lines correspond to different determinations of $k_1(\omega)$. The points with Im$[N(\omega)]=0$ correspond to solutions of the problem.

V. CONCLUDING REMARKS

We have discussed the changes (metamorphosis) of the spectrum of monochromatic modes in a semiconductor laser with optical feedback as a function of the reflectivity of the external mirror.

The changes observed belong to two classes, the first class correspond to the formation of ''islands'' as a consequence of coupling the laser to the external cavity. This change occurs for very low values of the reflectivity and is qualitatively similar to those described by the Lang-Kobayashi equations [2,3].

The second change involves the boundary conditions more deeply and is related to bifurcations of the solutions of the boundary condition equations and to the fact that they are multivalued equations. This second change establishes a limit of validity of the Lang-Kobayashi equations as R

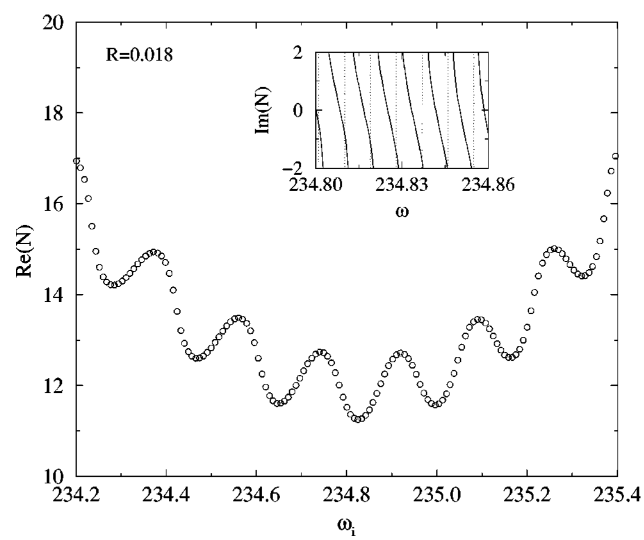

FIG. 8. Carrier density vs frequency for reflectivity $R=0.018$. In the inset the function Im$[N(\omega)]$. The modes coming from the inverse algorithm (15) have a very large carrier density and are not shown.

$\sim$0.011 [a value that is expected to depend on the modeling of the semiconductor susceptibility $\chi(\omega)$ but very weakly on the position of the external mirror]. Beyond this value of reflectivity, the description of the laser with optical feedback requires a multilongitudinal model such as the one introduced in [11].

It is important to realize that the need of a multilongitudinal model has been already suggested at least in one experimental work [4] where multilongitudinal operation of a semiconductor laser with optical feedback has been reported beyond the region known as coherence collapse.

ACKNOWLEDGMENTS

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[12] For the mathematically inclined readership we add the following: As a consequence of the merging of images, the number of times the preimage circle, $|W|=R$, is traveled changes when the image is circulated once, i.e., there are always n turns around $W=0$ needed to travel once the image of $|W|=R$ when this image encloses n images of $W=0$ (the relation is a consequence of the theorem of complex analysis known as the argument principle).