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The Heaviside–Feynman expression for the fields of an accelerated dipole

J. J. MONAGHAN
Churchill College, University of Cambridge, Cambridge
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Abstract. The fields due to an accelerated dipole are calculated using straightforward electromagnetic theory. It is shown that the potentials and fields may be written so that the static field, its first correction and the radiation field are separated. The expression for the fields and potentials is therefore analogous to the expression for the field of a point charge discovered by Heaviside and Feynman.

1. Introduction

Despite the fact that there exist in nature many objects which have magnetic or electric dipole moments it is customary when considering electromagnetic radiation to focus attention on the effects of accelerated point charges. The reason for this, of course, is that the most common source of radiation is the electron which, though it possesses a magnetic moment, can be considered a point charge with little error except for radiation of such high frequency that quantum effects become important. The few cases that arise where a more elaborate treatment is necessary are essentially examples of currents in a fixed region of space changing with time. These cases can be considered as simple generalizations of the formulae for the point charge, and therefore give no information about the radiation from dipoles moving in an arbitrary manner. This latter problem becomes of special importance in astrophysics where moving magnetic stars, for example, may give rise to considerable radiation.

Our concern here, however, will not be with the detailed application of the theory but with the derivation of expressions for the potentials and fields of moving dipoles. In recent years Ward (1964, 1965) and Ellis (1963, 1966) have examined the fields due to accelerated dipoles, and Ellis, in particular, has obtained expressions for the radiation field. The final results we shall present are, therefore, for the most part not new. However, the methods employed by Ward (1964, 1965) and Ellis (1963) are rather complicated; for example, Ellis follows the world lines of single poles which are finally compounded to yield the dipole, while Ward employs a special method of integrating Maxwell's equations. Both methods lead to complex expressions devoid of a simple physical interpretation. In his most recent work Ellis (1966) uses an invariant form throughout, reaching a succinct formulation of the fields in a concise manner related to that to be described here. However, when the succinct form is unfolded for application Ellis recovers the complicated expression he derived earlier. For this reason the work presented here may be of interest since we shall derive the results easily using well-known methods. In addition we shall show how the potentials for an accelerated particle with a pure magnetic dipole in its rest frame can be written analogously to the beautiful expression for the fields of a point charge first discovered by Heaviside (1902) and rediscovered by Feynman (1950).

2. The point charge

To place our results in perspective we first derive the fields of the point charge in the Heaviside–Feynman form. Let the charge of the particle be e, its velocity $\mathbf{v}$ and $\mathbf{R}$ the vector from the observer at $\mathbf{x}$ to the field point $\mathbf{x}'$. If the position of the particle is given by $\mathbf{r}(t)$ then the current four-vector is

$$J_\mu = ec\beta_\mu \delta\{\mathbf{x}' - \mathbf{r}(t)\} \tag{2.1}$$

where $\beta_\mu = (1, \boldsymbol{\beta})$ and $\boldsymbol{\beta} = \mathbf{v}/c$. The potential four-vector is then

$$A_\mu = e \int\int \frac{\beta_\mu}{R} \delta\left(t' + \frac{R}{c} - t\right) \delta\{\mathbf{x}' - \mathbf{r}(t')\}\, dt'\, d^3x' \tag{2.2}$$

the retarded behaviour being provided by $\delta(t'+R/c-t)$. The volume integral in (2.2) can be done immediately, resulting in

$$A_\mu = e\int \frac{\beta_\mu}{R}\delta\left(t'+\frac{R}{c}-t\right)dt'. \tag{2.3}$$

If desired the integration over t' could also be performed, giving the classical Lienard Wiechert potentials. However, the determination of the fields is easier if the integral form is retained. We now calculate $\mathbf{E}$ from

$$\mathbf{E} = -\frac{1}{c}\frac{\partial \mathbf{A}}{\partial t} - \nabla\Phi$$

by differentiating under the integral sign in (2.3), noting that $\partial/\partial t$ operates only on δ and ∇ operates only on R. We find

$$\mathbf{E} = e\int\left\{\frac{\mathbf{n}\delta}{R^2}-\frac{1}{Rc}(\boldsymbol{\beta}-\mathbf{n})\frac{d\delta}{dt'}\right\}dt' \tag{2.4}$$

$$= e\left\{\frac{\mathbf{n}}{KR^2}+\frac{1}{c}\frac{d}{dt}\left(\frac{\mathbf{n}-\boldsymbol{\beta}}{KR}\right)\right\}_{\text{ret}}. \tag{2.5}$$

Here $\mathbf{n}$ is a unit vector in the direction $\mathbf{R}$ and ret denotes that the bracketed quantity is to be evaluated at the retarded time $t' = t-R/c$. The factor K is defined by

$$K = \frac{dt}{dt'} = 1+\frac{1}{c}\frac{dR}{dt'} = 1-\mathbf{n}.\boldsymbol{\beta}. \tag{2.6}$$

According to this definition

$$\frac{1}{K} = 1-\frac{1}{c}\frac{dR}{dt} \tag{2.7}$$

an expression we shall use frequently. Substituting for $1/K$ in (2.5) and using the fact that

$$\boldsymbol{\beta} = -\frac{1}{c}\frac{d}{dt'}(R\mathbf{n}) \tag{2.8}$$

(2.5) becomes

$$\mathbf{E} = e\left[\frac{\mathbf{n}}{R^2}\left(1-\frac{1}{c}\frac{dR}{dt}\right)+\frac{1}{c}\frac{d}{dt}\left\{\frac{\mathbf{n}}{R}\left(1-\frac{1}{c}\frac{dR}{dt}\right)+\frac{1}{Rc}\frac{d}{dt}(R\mathbf{n})\right\}\right]_{\text{ret}}$$

$$= e\left\{\frac{\mathbf{n}}{R^2}+\frac{R}{c}\frac{d}{dt}\left(\frac{\mathbf{n}}{R^2}\right)+\frac{1}{c^2}\frac{d^2\mathbf{n}}{dt^2}\right\}_{\text{ret}}. \tag{2.9}$$

This is the Heaviside–Feynman formula. To calculate $\mathbf{B}$ we proceed as before using $\mathbf{B} = \nabla \wedge \mathbf{A}$. We find

$$\mathbf{B} = \left\{\frac{\boldsymbol{\beta}\wedge\mathbf{n}}{KR^2}+\frac{d}{dt}\left(\frac{\boldsymbol{\beta}\wedge\mathbf{n}}{KcR}\right)\right\}_{\text{ret}}. \tag{2.10}$$

But

$$\frac{\boldsymbol{\beta}\wedge\mathbf{n}}{K} = -\frac{R}{c}\frac{d\mathbf{n}}{dt}\wedge\mathbf{n}$$

so (2.10) becomes

$$\mathbf{B} = e\left(\frac{\mathbf{n}}{Rc}\wedge\frac{d\mathbf{n}}{dt}+\frac{1}{c^2}\mathbf{n}\wedge\frac{d^2\mathbf{n}}{dt^2}\right)_{\text{ret}} \tag{2.11}$$

or $\mathbf{B} = \mathbf{n}\wedge\mathbf{E}$, as expected.

Apart from the simplicity of (2.9) it is especially useful because each term may be given a physical interpretation. Thus the first term gives the retarded static field while the second term gives the first-order correction for the retardation. The direct static field is therefore a better approximation to the field of a moving charge than one would at first suppose. The third term contains all the radiation effects and provides the simplest starting point for a determination of the frequency spectra of the radiation. A very nice discussion of the interpretation of (2.9) is given by Feynman in the reference cited.

3. The dipole potentials

To obtain the fields of a moving dipole we can proceed as before. To establish the form of the current four-vector it is convenient to consider the dipole as having a polarization density $\mathbf{P}(t')$ and a magnetization density $\mathbf{M}(t')$ as seen in the observer's frame. The observer will then see a charge and current density

$$\rho = -\nabla.\mathbf{P} \tag{3.1}$$

$$\mathbf{J} = c\nabla \wedge \mathbf{M} + \frac{\partial \mathbf{P}}{\partial t'} \tag{3.2}$$

which follow from standard results in electromagnetic theory (Jackson 1962, p. 198). The current four-vector $(c\rho, \mathbf{J})$ is thus

$$J = \left(-c\nabla.\mathbf{P}, \frac{\partial \mathbf{P}}{\partial t'} + c\nabla \wedge \mathbf{M}\right). \tag{3.3}$$

Although it is not necessary for what follows, it is useful to note that (3.3) can be written in a covariant way by defining a skew symmetric tensor $P_{\mu\nu}$ such that

$$J_\nu = \nabla_\mu P_{\mu\nu}$$

where ∇_μ is the gradient four-vector $((1/c)\partial/\partial t', -\nabla)$ and $P_{xy} = -M_z$, $P_{yz} = -M_x$, $P_{xt} = -P_x$, $P_{yt} = -P_y$, etc.

It is important to realize that $\mathbf{P}$ and $\mathbf{M}$ are densities, and the appropriate tensor expressions for the electromagnetic dipole moment tensor is $P_{\mu\nu}\, dv_0$ where dv_0 is an element of proper volume. If one integrates over the volume in the observer's frame then a factor of $\sqrt{(1-v^2/c^2)}$ is introduced.

Having found $\mathbf{J}$ and ρ we can proceed as before, except we find it convenient here to split A_μ into its components $(\Phi, \mathbf{A})$. Thus

$$\mathbf{A} = \int\int \left(\frac{1}{c}\frac{\partial \mathbf{P}}{\partial t'} + \nabla \wedge \mathbf{M}\right) \frac{\delta(t' + R/c - t)}{R}\, dt'\, d^3x' \tag{3.4}$$

$$\Phi = -\int\int \frac{\nabla.\mathbf{P}}{R}\, \delta\left(t' + \frac{R}{c} - t\right) dt'\, d^3x'. \tag{3.5}$$

For the point dipole we have

$$\begin{aligned} \mathbf{P} &= \mathbf{p}\delta\{\mathbf{x}' - \mathbf{r}(t')\} \\ \mathbf{M} &= \boldsymbol{\mu}\delta\{\mathbf{x}' - \mathbf{r}(t')\}. \end{aligned} \tag{3.6}$$

The vector potential can be evaluated in the following way. The first term can be integrated by parts and the second simplified using the result

$$\int f\nabla \wedge \{\boldsymbol{\mu}\delta(\mathbf{x})\}\, d^3x = \boldsymbol{\mu} \wedge \nabla f. \tag{3.7}$$

Thus

$$\mathbf{A} = \int \left\{-\frac{\mathbf{p}\delta'}{cR} + \boldsymbol{\mu} \wedge \left(\frac{\mathbf{n}\delta}{R^2} - \frac{\mathbf{n}\delta'}{Rc}\right)\right\} dt' \tag{3.8}$$

where δ' denotes a derivative of the delta function with respect to its argument. If we perform the final integration in (3.8), we obtain

$$\mathbf{A} = \left\{\frac{\boldsymbol{\mu}\wedge\mathbf{n}}{KR^2}+\frac{d}{dt}\left(\frac{\boldsymbol{\mu}\wedge\mathbf{n}}{KRc}\right)+\frac{d}{dt}\left(\frac{\mathbf{p}}{KRc}\right)\right\}_{\text{ret}}. \tag{3.9}$$

Proceeding in a similar way, we find for Φ the expression

$$\Phi = \left\{\frac{\mathbf{p}.\mathbf{n}}{KR^2}+\frac{d}{dt}\left(\frac{\mathbf{p}.\mathbf{n}}{KRc}\right)\right\}_{\text{ret}}. \tag{3.10}$$

If one requires the electric and magnetic fields, (3.9) and (3.10) are not the best to work with since differentiations are best carried out before the integrals are evaluated. However, they can be written in an interesting way by using the expression (2.7) for $1/K$. Substituting for $1/K$ and rearranging terms we find

$$\mathbf{A} = \left\{\frac{\boldsymbol{\mu}\wedge\mathbf{n}}{R^2}+\frac{R}{c}\frac{d}{dt}\left(\frac{\boldsymbol{\mu}\wedge\mathbf{n}}{R^2}\right)+\frac{d}{dt}\left(\frac{\mathbf{p}}{KRc}-\frac{\boldsymbol{\mu}\wedge\mathbf{n}}{Rc^2}\frac{dR}{dt}\right)\right\}_{\text{ret}}. \tag{3.11}$$

Similarly, one finds for Φ the expression

$$\Phi = \left\{\frac{\mathbf{p}.\mathbf{n}}{R^2}+\frac{R}{c}\frac{d}{dt}\left(\frac{\mathbf{p}.\mathbf{n}}{R^2}\right)-\frac{1}{c^2}\frac{d}{dt}\left(\frac{\mathbf{p}.\mathbf{n}}{R}\frac{dR}{dt}\right)\right\}_{\text{ret}}. \tag{3.12}$$

If the moment is a pure magnetic dipole in the rest frame of the particle one finds by using the transformation rules for $\mathbf{p}$ and $\boldsymbol{\mu}$ that $\mathbf{p} = \boldsymbol{\beta}\wedge\boldsymbol{\mu}$. Substituting this result in (3.11) we find on using (2.8)

$$\mathbf{A} = \left\{\frac{\boldsymbol{\mu}\wedge\mathbf{n}}{R^2}+\frac{R}{c}\frac{d}{dt}\left(\frac{\boldsymbol{\mu}\wedge\mathbf{n}}{R^2}\right)+\frac{1}{c^2}\frac{d}{dt}\left(\boldsymbol{\mu}\wedge\frac{d\mathbf{n}}{dt}\right)\right\}_{\text{ret}} \tag{3.13}$$

with a similar, but not quite so compact expression for Φ. This expression is almost exactly the same as (2.9) and may be interpreted similarly except for the last term which is not the sole contributor to the radiation. The reader may find it useful to compare (3.13) with the result due to Ellis (1963, p. 765).

4. The dipole fields

To obtain $\mathbf{E}$ we return to the integral expressions for $\mathbf{A}$ and Φ and use the relation

$$\mathbf{E} = -\frac{1}{c}\frac{\partial\mathbf{A}}{\partial t}-\nabla\Phi.$$

The calculation is straightforward and we find

$$\frac{1}{c}\frac{\partial\mathbf{A}}{\partial t} = \left\{\frac{d}{dt}\left(\frac{\boldsymbol{\mu}\wedge\mathbf{n}}{R^2Kc}\right)+\frac{d^2}{dt^2}\left(\frac{\boldsymbol{\mu}\wedge\mathbf{n}}{KRc^2}\right)+\frac{d^2}{dt^2}\left(\frac{\mathbf{p}}{KRc^2}\right)\right\}_{\text{ret}}$$

$$-\nabla\Phi = \left\{\frac{3(\mathbf{p}.\mathbf{n})\mathbf{n}-\mathbf{p}}{KR^3}+\frac{d}{dt}\left(\frac{3(\mathbf{p}.\mathbf{n})\mathbf{n}-\mathbf{p}}{KR^2c}\right)+\frac{d^2}{dt^2}\left(\frac{\mathbf{n}(\mathbf{p}.\mathbf{n})}{Kc^2R}\right)\right\}_{\text{ret}}.$$

Therefore

$$\mathbf{E} = \left[\frac{3(\mathbf{p}.\mathbf{n})\mathbf{n}-\mathbf{p}}{KR^3}+\frac{d}{dt}\left\{\frac{3(\mathbf{p}.\mathbf{n})\mathbf{n}-\mathbf{p}}{KR^2c}\right\}-\frac{d}{dt}\left(\frac{\boldsymbol{\mu}\wedge\mathbf{n}}{KR^2c}\right)\right.$$
$$\left.+\frac{d^2}{dt^2}\left(\frac{\mathbf{n}\wedge(\mathbf{n}\wedge\mathbf{p})-\boldsymbol{\mu}\wedge\mathbf{n}}{KRc^2}\right)\right]_{\text{ret}}. \tag{4.1}$$

To cast this in a form similar to (2.9) we substitute for K. We find

$$\mathbf{E} = \left[\frac{3(\mathbf{p}.\mathbf{n})\mathbf{n}-\mathbf{p}}{R^3}+\frac{R}{c}\frac{d}{dt}\left\{\frac{3(\mathbf{p}.\mathbf{n})\mathbf{n}-\mathbf{p}}{R^3}\right\}\right.$$
$$\left.-\frac{d}{dt}\left\{\frac{3(\mathbf{p}.\mathbf{n})\mathbf{n}-\mathbf{p}}{c^2R^2}\frac{dR}{dt}+\frac{\boldsymbol{\mu}\wedge\mathbf{n}}{KcR^2}\right\}-\frac{1}{c^2}\frac{d^2}{dt^2}\left\{\frac{\mathbf{n}\wedge(\mathbf{n}\wedge\mathbf{p})-\boldsymbol{\mu}\wedge\mathbf{n}}{KR}\right\}\right]_{\text{ret}}. \tag{4.2}$$

Here, as before, the first two terms represent the 'corrected' retarded static field. The last term contains all the radiation effects and is therefore analogous to the $(1/c^2)\, d^2n/dt^2$ term in (2.9). The third term is an enigma and despite considerable effort no satisfactory physical interpretation of it has been found. It can, however, be simplified in the special case where $\boldsymbol{\mu} = 0$ in the rest frame of the particle. For this case $\boldsymbol{\mu} = \mathbf{p} \wedge \boldsymbol{\beta}$ and the term becomes

$$-\frac{1}{c^2}\frac{d}{dt}\left\{\mathbf{p}.\mathbf{n}R\frac{d}{dt}\left(\frac{\mathbf{n}}{R^2}\right)\right\}.$$

By making use of the fact that $\mathbf{B} = \nabla \wedge \mathbf{A}$ we can easily obtain a similar expression for $\mathbf{B}$. Omitting the details, which are similar to those involved in calculating $\mathbf{E}$, we find

$$\mathbf{B} = \left[\frac{3(\boldsymbol{\mu}.\mathbf{n})\mathbf{n}-\boldsymbol{\mu}}{R^3}+\frac{R}{c}\frac{d}{dt}\left\{\frac{3(\boldsymbol{\mu}.\mathbf{n})\mathbf{n}-\boldsymbol{\mu}}{R^3}\right\}\right.$$
$$\left.-\frac{d}{dt}\left\{\frac{3(\boldsymbol{\mu}.\mathbf{n})\mathbf{n}-\boldsymbol{\mu}}{c^2R^2}\frac{dR}{dt}-\frac{\mathbf{p}\wedge\mathbf{n}}{KR^2c}\right\}+\frac{1}{c^2}\frac{d^2}{dt^2}\left\{\frac{\mathbf{n}\wedge(\mathbf{n}\wedge\boldsymbol{\mu})+\mathbf{p}\wedge\mathbf{n}}{KR}\right\}\right]_{\text{ret}}. \tag{4.3}$$

Reference to (4.2) shows the symmetry between the expressions. One may easily show from (4.2) and (4.3) that the radiation fields satisfy $\mathbf{B} = \mathbf{n} \wedge \mathbf{E}$ since $\mathbf{n}$ is effectively constant at large distances. Again the third term is difficult to interpret, but it can be simplified if $\mathbf{p} = 0$ in the rest frame. We find it becomes

$$\frac{1}{c^2}\frac{d}{dt}\left\{\boldsymbol{\mu}.\mathbf{n}R\frac{d}{dt}\left(\frac{\mathbf{n}}{R^2}\right)\right\}.$$

Finally, we note that in the particular case where $\boldsymbol{\mu} = 0$ in the rest frame of the particle we can write the radiation term in $\mathbf{E}$ as

$$\frac{1}{Rc^2}\frac{d^2}{dt^2}\left[\mathbf{n}\wedge\left\{\mathbf{n}\wedge\left(\mathbf{p}+\frac{\mathbf{p}.\mathbf{n}}{K}\boldsymbol{\beta}\right)\right\}\right]. \tag{4.4}$$

This is the same expression as that obtained by Ward (1964).

5. Radiated power

As we noted earlier, Ellis has obtained the expressions for the radiated power using his form of the radiation fields. Here we give an example to show how the power can be calculated quickly using our representation of $\mathbf{E}$. We suppose $\boldsymbol{\mu} = 0$ in the rest frame of the particle and assume in addition that the dipole does not rotate or oscillate but moves in a circular orbit with $\mathbf{p}$ parallel to $\boldsymbol{\beta}$. We choose a spherical polar coordinate system with $\boldsymbol{\beta}$ in the direction $\theta = 0$, θ the angle between $\mathbf{n}$ and $\boldsymbol{\beta}$, and ϕ the angle between the plane containing $(\mathbf{n}, \boldsymbol{\beta})$ and that containing $(\boldsymbol{\beta}, d\boldsymbol{\beta}/dt')$. Here the radiation field is

$$\mathbf{E} = \frac{1}{Rc^2K}\mathbf{n}\wedge\left[\mathbf{n}\wedge\frac{d}{dt'}\left\{\frac{1}{K}\frac{d}{dt'}\left(\frac{\mathbf{p}}{K}\right)\right\}\right]. \tag{5.1}$$

After differentiation we find

$$\mathbf{E} = \frac{1}{Rc^2K^4}\mathbf{n}\wedge\left[\mathbf{n}\wedge\left\{\mathbf{p}\left(\frac{3\dot{K}^2}{K}-\ddot{K}\right)-3\dot{K}\frac{d\mathbf{p}}{dt'}+K\frac{d^2\mathbf{p}}{dt'^2}\right\}\right] \tag{5.2}$$

where dots denote derivatives with respect to t'.

For this case

$$\mathbf{p} = \frac{p_0}{\gamma}\frac{\boldsymbol{\beta}}{\beta} \qquad \frac{d\mathbf{p}}{dt'} = \frac{p_0}{\gamma\beta}\frac{d\boldsymbol{\beta}}{dt'} \qquad \frac{d^2\mathbf{p}}{dt'^2} = -\frac{\boldsymbol{\beta}}{\gamma\beta}\left(\frac{\beta'^2}{\beta^2}\right)$$

where γ (equal to $1/\sqrt{(1-v^2/c^2)}$) provides the relativistic contraction but is itself unchanged by the motion. Substituting in (5.2) we find

$$\mathbf{E} = \frac{p_0\dot{\beta}^2}{\gamma R c^2 K^4}\left\{\mathbf{n}\wedge(\mathbf{n}\wedge\mathbf{k})\left(\frac{3\sin^2\theta\cos^2\phi}{K}-\frac{1}{\beta^2}\right)+\frac{3\sin\theta\cos\phi}{\beta}\mathbf{n}\wedge(\mathbf{n}\wedge\mathbf{i})\right\}$$

where $\mathbf{R} = \boldsymbol{\beta}/\beta$ and $\mathbf{i} = (d\boldsymbol{\beta}/dt')/|\dot{\beta}|'$.

In terms of the particle's own time the power radiated per unit solid angle is

$$\frac{dP}{d\Omega} = \frac{cK}{4\pi}|\mathbf{E}|^2R^2.$$

On subsituting for E we finally obtain

$$\frac{dP}{d\Omega} = \frac{\dot{\beta}^4 p_0{}^2\sin^2\theta}{4\pi\gamma^2c^3K^9\beta^4}\left[\{3\cos^2\phi(\beta^2-1+K)-K\}^2+\left(\frac{3K\beta\sin 2\phi}{2}\right)^2\right].$$

This is Ellis's result. If $\mathbf{p}$ is perpendicular to $\boldsymbol{\beta}$ the use of (4.4) shows that the power is a multiple of the foregoing expression.

If the power spectrum is needed then the electric field as presented in (4.2) is particularly easy to use since the second derivative allows two immediate integrations by parts. The reader will be able to work out the details readily.

6. Conclusion

We have been able to derive comparatively simple vector expressions for the potentials and fields of a dipole moving in any manner. The resultant expression for the vector potential consists of terms which represent successively the retarded static potential, a first-order 'correction' to this potential and a third term arising from the acceleration. In this form the potential is equivalent to the expression for the electric field for a moving point charge discovered by Heaviside and Feynman. The electric and magnetic fields are similar in form except for an additional term not associated with radiation. The radiation term, as in the case of the point charge, can be represented as a second derivative and is therefore particularly useful for power spectra calculations.

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References

Ellis, J. R., 1963, *Proc. Camb. Phil. Soc.*, **59**, 759–74.
—— 1966, *J. Math. Phys.*, **7**, 1185–97.
Feynman, R. P., 1964, *Lectures on Physics*, Vol. II (Reading, Mass.: Addison-Wesley).
Heaviside, O., 1912, *Electromagnetic Theory*, Vol. III (London: The Electrician Printing and Publishing Co.).
Jackson, J. D., 1962, *Classical Electrodynamics* (London: John Wiley).
Ward, G. N., 1964, *Proc. R. Soc.* A, **279**, 562–71.
—— 1965, *Proc. Camb. Phil. Soc.*, **61**, 547–53.