Radiation fields of a dipole in arbitrary motion

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We present a version of Jefimenko's formulas for electric and magnetic fields in material media, and we demonstrate how this version of the formulas may be used for deriving the electric and magnetic fields produced by an arbitrarily moving dipole containing both electric and magnetic moments. Like the fields produced by a charge in arbitrary motion, the fields of an arbitrarily moving dipole can also be written in two forms: The Heaviside–Feynman form and the Liénard–Wiechert form. We derive here the first form and the radiation fields associated with the second form. These radiation fields are expressed by means of concise, symmetric, and illuminating formulas.

I. INTRODUCTION

Jefimenko's remarkable contribution[1] that the usual retarded solutions of Maxwell's equations may be expressed without spatial derivatives has generated a great deal of interest[2–6] (the spatial derivatives are troublesome because of their subtle action on both field and source coordinates hidden inside the *retarded time*). Recently Jefimenko[7] himself has extended his approach (by which, roughly speaking, the spatial derivatives of retarded quantities may be transformed into time derivatives) to the electric and magnetic fields in material media. However, his extended formulas [Eqs. (1) and (2) of the present paper] still contain some spatial derivatives.

In this paper we present a version of Jefimenko's formulas in material media which contains no spatial derivatives (Sec. II). We then use this version of the formulas for discussing an interesting but complicated problem: To find the electric and magnetic fields of a dipole in arbitrary motion. Like the fields of a point charge in arbitrary motion, the fields produced by an arbitrarily moving dipole (containing both electric and magnetic moments) can be written in two forms: The Heaviside–Feynman form and the Liénard–Wiechert form. While the first form has been previously derived by means of the usual method of potentials,[8] the second form has been discussed (in terms of unconventional parameters) by means of specialized methods.[9–12] In this paper we demonstrate how our version of Jefimenko's formulas can be used to derive the first form (Sec. III), and the radiation fields associated with the second form (Sec. IV). These last fields are expressed by means of concise, symmetric, and illuminating formulas. A complementary Appendix closes the paper.

Due to the importance of dipoles—many fundamental objects in the nature have magnetic or electric dipole moments—we have chosen them to illustrate the power of Jefimenko's approach, and also to provide a useful reference for those students who inquire about what happens when a dipole moves arbitrarily.

II. A REFORMULATION OF JEFIMENKO'S FORMULAS IN MATERIAL MEDIA

Jefimenko[7] has recently extended his formulas for electric and magnetic fields, **E** and **B**, in material media (in which the free charge density ρ, the free current density **J**, the polarization **P**, and magnetization **M** are all specified quantities). These formulas can be written as

$$\mathbf{E}=\frac{1}{4\pi\epsilon_0}\int\left\{\frac{[\rho]\hat{r}}{r^2}+\frac{[\partial\rho/\partial t]\hat{r}}{rc}-\frac{[\partial\mathbf{J}/\partial t]}{rc^2}\right\}dv' -\frac{1}{4\pi}\frac{\partial}{\partial t}\int\frac{[\nabla'\times\mathbf{M}]}{r}dv'-\frac{1}{4\pi\epsilon_0}\int\left\{\frac{[\nabla'\cdot\mathbf{P}]\hat{r}}{r^2} +\frac{(\partial[\nabla'\cdot\mathbf{P}]/\partial t)\hat{r}}{rc}\right\}dv'-\frac{1}{4\pi\epsilon_0}\int\frac{[\partial^2\mathbf{P}/\partial t^2]}{rc^2}dv', \tag{1}$$

$$\mathbf{B}=\frac{\mu_0}{4\pi}\int\left\{\frac{[\mathbf{J}]\times\hat{r}}{r^2}+\frac{[\partial\mathbf{J}/\partial t]\times\hat{r}}{rc}\right\}dv' +\frac{\mu_0}{4\pi}\frac{\partial}{\partial t}\int\frac{[\nabla'\times\mathbf{P}]}{r}dv'+\frac{1}{4\pi}\int\left\{\frac{[\nabla'\times\mathbf{M}]\times\hat{r}}{r^2} +\frac{(\partial[\nabla'\times\mathbf{M}]/\partial t)\times\hat{r}}{rc}\right\}dv', \tag{2}$$

where the integrals are taken over all space; with **r** the *field point* (at which the electric field **E** and the magnetic field **B** are evaluated) and $\mathbf{r}'$ the *source point* (at which ρ, **J**, $\nabla\cdot\mathbf{P}$, $\nabla\times\mathbf{P}$, and $\nabla\times\mathbf{M}$ are evaluated), $r=|\mathbf{r}-\mathbf{r}'|$ and $\hat{r}=(\mathbf{r}-\mathbf{r}')/r$; and finally, the square brackets $[\cdots]$ denote *retardation*, indicating that the bracketed quantity is to be evaluated at the *source point* $\mathbf{r}'$ and *retarded time* $t'=t-r/c$, where t is the time for which the **E** and **B** fields are evaluated, and c is the speed of light. For example,

$$[\mathbf{J}]=\mathbf{J}(\mathbf{r}',t')=\mathbf{J}(\mathbf{r}',t-r/c). \tag{3}$$

In stating Eqs. (1) and (2),[13] Jefimenko has defined the magnetization **M** by the unusual expression $\mathbf{M}=\mathbf{B}-\mu_0\mathbf{H}$ (recall that the usual definition of **M** is $\mathbf{M}=\mathbf{B}/\mu_0-\mathbf{H}$). Jefimenko's unconventional placement of the μ_0 factor in the definition of **M** will imply a convenient symmetry between the **E** and **B** fields later in this paper.

As may be seen, Eqs. (1) and (2) still have spatial-derivative operations ("∇ operations") inside the retardation symbol. These ∇ operations make it awkward to apply Eqs. (1) and (2). The action of these operations on both field and source coordinates hidden inside the *retarded time* is subtle. Following Jefimenko, we shall remove these ∇ operations by recasting them into time derivatives. More specifically we are going to transform Eqs. (1) and (2) in such a way that they both contain no spatial derivatives. Consider first the following relations (which are proved in the Appendix):

$$\frac{\partial}{\partial t}\int\frac{[\nabla'\times\mathbf{M}]}{r}\,dv'$$
$$=\int\left\{\frac{[\partial\mathbf{M}/\partial t]\times\hat{r}}{r^2}+\frac{[\partial^2\mathbf{M}/\partial t^2]\times\hat{r}}{rc}\right\}dv', \qquad (4)$$

$$\int\left\{\frac{[\nabla'\cdot\mathbf{P}]\hat{r}}{r^2}+\frac{(\partial[\nabla'\cdot\mathbf{P}]/\partial t)\hat{r}}{rc}\right\}dv'$$
$$=-\int\left\{\frac{3([\mathbf{P}]\cdot\hat{r})\hat{r}-[\mathbf{P}]}{r^3}+\frac{3([\partial\mathbf{P}/\partial t]\cdot\hat{r})\hat{r}-[\partial\mathbf{P}/\partial t]}{r^2c}+\frac{([\partial^2\mathbf{P}/\partial t^2]\cdot\hat{r})\hat{r}}{rc^2}\right\}dv'. \qquad (5)$$

Applying Eqs. (4) and (5) to Eq. (1), the electric field **E** takes the form:

$$\mathbf{E}=\frac{1}{4\pi\epsilon_0}\int\left\{\frac{[\rho]\hat{r}}{r^2}+\frac{[\partial\rho/\partial t]\hat{r}}{rc}-\frac{[\partial\mathbf{J}/\partial t]}{rc^2}\right\}dv'$$
$$-\frac{1}{4\pi}\int\left\{\frac{[\partial\mathbf{M}/\partial t]\times\hat{r}}{r^2}+\frac{[\partial^2\mathbf{M}/\partial t^2]\times\hat{r}}{rc}\right\}dv'$$
$$+\frac{1}{4\pi\epsilon_0}\int\left\{\frac{3([\mathbf{P}]\cdot\hat{r})\hat{r}-[\mathbf{P}]}{r^3}+\frac{3([\partial\mathbf{P}/\partial t]\cdot\hat{r})\hat{r}-[\partial\mathbf{P}/\partial t]}{r^2c}+\frac{([\partial^2\mathbf{P}/\partial t^2]\cdot\hat{r})\hat{r}-[\partial^2\mathbf{P}/\partial t^2]}{rc^2}\right\}dv'. \qquad (6)$$

By a similar procedure, we can transform the magnetic field in Eq. (2). Applying Eq. (4) with **P** instead of **M**, and the following relation (see the Appendix):

$$\int\left\{\frac{[\nabla'\times\mathbf{M}]\times\hat{r}}{r^2}+\frac{\partial[\nabla'\times\mathbf{M}]/\partial t\times\hat{r}}{rc}\right\}dv'$$
$$=\int\left\{\frac{3([\mathbf{M}]\cdot\hat{r})\hat{r}-[\mathbf{M}]}{r^3}+\frac{3([\partial\mathbf{M}/\partial t]\cdot\hat{r})\hat{r}-[\partial\mathbf{M}/\partial t]}{r^2c}+\frac{([\partial^2\mathbf{M}/\partial t^2]\cdot\hat{r})\hat{r}-[\partial^2\mathbf{M}/\partial t^2]}{rc^2}\right\}dv', \qquad (7)$$

to Eq. (2), we obtain

$$\mathbf{B}=\frac{\mu_0}{4\pi}\int\left\{\frac{[\mathbf{J}]\times\hat{r}}{r^2}+\frac{[\partial\mathbf{J}/\partial t]\times\hat{r}}{rc}\right\}dv'$$
$$+\frac{\mu_0}{4\pi}\int\left\{\frac{[\partial\mathbf{P}/\partial t]\times\hat{r}}{r^2}+\frac{[\partial^2\mathbf{P}/\partial t^2]\times\hat{r}}{rc}\right\}dv'$$
$$+\frac{1}{4\pi}\int\left\{\frac{3([\mathbf{M}]\cdot\hat{r})\hat{r}-[\mathbf{M}]}{r^3}+\frac{3([\partial\mathbf{M}/\partial t]\cdot\hat{r})\hat{r}-[\partial\mathbf{M}/\partial t]}{r^2c}+\frac{([\partial^2\mathbf{M}/\partial t^2]\cdot\hat{r})\hat{r}-[\partial^2\mathbf{M}/\partial t^2]}{rc^2}\right\}dv'. \qquad (8)$$

The formulas (6) and (8) constitute our version of Jefimenko's formulas in material media. At first sight, it might not appear very fruitful to consider Eqs. (6) and (8) instead of Eqs. (1) and (2), since the former contain more terms than the latter. However, Eqs. (6) and (8) are more convenient than Eqs. (1) and (2) in the sense that they contain no spatial derivatives, which are made cumbersome by retardation. Obviously Eqs. (6) and (8) reduce more transparently to the familiar static forms[14] and they are more symmetric than Eqs. (1) and (2).

Formulas (6) and (8) permit us to solve easily the well-known problem of calculating the fields **E** and **B** due to a point oscillating electric dipole, i.e., the so-called problem of the *Hertzian* dipole.[15] Similarly we can solve the analogous problem for a magnetic dipole.[16] Furthermore, Eqs. (6) and (8) permit us to calculate the **E** and **B** fields of an oscillating dipole with dual moment, i.e., one containing both electric and magnetic moments. Indeed, consider such a *generalized* point dipole which contains both electric moment $\mathbf{p}=p\mathbf{e}_z$ and magnetic moment $\mathbf{m}=m\mathbf{e}_z$ ($\mathbf{e}_z$ is a unit vector in the Z direction) and is located at the origin of coordinates. Both p and m are functions of the retarded time: $p(t')=p_0\exp(-i\omega t')$ and $m(t')=m_0\exp(-i\omega t')$. The polarization and magnetization have a value only at the position of the dipole, i.e., at the origin

$$[\mathbf{P}]=\mathbf{P}(\mathbf{r}',t')=\mathbf{p}(t')\delta^3(\mathbf{r}'), \qquad (9)$$

$$[\mathbf{M}]=\mathbf{M}(\mathbf{r}',t')=\mathbf{m}(t')\delta^3(\mathbf{r}'), \qquad (10)$$

where δ^3 is the three-dimensional Dirac delta function. For these specific sources, the formulas (6) and (8) involve simple integrations, e.g.,

$$\frac{\partial}{\partial t}\int\frac{[\mathbf{M}]}{r}\,dv'=\frac{[\dot{\mathbf{m}}]}{r}, \qquad (11)$$

where now $r=|\mathbf{r}|$ since $\mathbf{r}'=0$ for the point dipole, and $\partial\mathbf{m}/\partial t\equiv\dot{\mathbf{m}}$. After performing all the corresponding integrations, the final result can be written as

$$\mathbf{E}=\left(\frac{1}{4\pi\epsilon_0}\right)\hat{r}\times\left(\hat{r}\times\left\{\frac{[\ddot{\mathbf{p}}]}{rc^2}+\frac{3[\dot{\mathbf{p}}]}{r^2c}+\frac{3[\mathbf{p}]}{r^3}\right\}+\epsilon_0\left\{\frac{[\ddot{\mathbf{m}}]}{rc}+\frac{[\dot{\mathbf{m}}]}{r^2}\right\}\right), \qquad (12)$$

$$\mathbf{B}=\left(\frac{1}{4\pi}\right)\hat{r}\times\left(\hat{r}\times\left\{\frac{[\ddot{\mathbf{m}}]}{rc^2}+\frac{3[\dot{\mathbf{m}}]}{r^2c}+\frac{3[\mathbf{m}]}{r^3}\right\}-\mu_0\left\{\frac{[\ddot{\mathbf{p}}]}{rc}+\frac{[\dot{\mathbf{p}}]}{r^2}\right\}\right). \qquad (13)$$

We find here two familiar cases: If $\mathbf{m}=0$ we have the fields of an oscillating electric dipole,[15] and if $\mathbf{p}=0$ we have the fields of an oscillating magnetic dipole.[16]

III. HEAVISIDE–FEYNMAN FORM OF THE FIELDS OF A DIPOLE IN ARBITRARY MOTION

In this section we shall illustrate the effectiveness our version of Jefimenko's formulas by solving an interesting but complicated problem: To determine the **E** and **B** fields due to an arbitrarily moving dipole which carries both **p** and **m** moments. This problem has been previously discussed by Monaghan[8] using the standard method of potentials. The problem is particularly important from a physical point of view because many objects in the nature have magnetic and/or electric dipole moments. A special case of this problem, namely, that in which the dipole carries exclusively **p** moment was treated by Ellis[9,12] and Ward.[10,11] To face this problem, both authors needed to develop methods quite specialized. This fact indicates the complexity of the problem and also explains why it was apparently not discussed until the 60's.[8–12] This is somewhat surprising, because, with exception of a point charge (for which the fields were determined a long time ago), a dipole is the simplest localized charge or current configuration.

Like the fields produced by an arbitrarily moving charge, the fields of a dipole in arbitrary motion can be expressed in two forms: The Heaviside–Feynman form and the Liénard–Wiechert form. Both forms have advantages and disadvantages. The structure of the first form is more compact but difficult to interpret, while the second form is easier to interpret but oppressively long in its structure.[17] In this section we will show how the first form may be derived easily from our Eqs. (6) and (8).

Let us consider a moving dipole with velocity $\mathbf{v}(t) = d\mathbf{s}(t)/dt$ at the point $\mathbf{s}(t)$. We assume that the dipole has polarization and magnetization which are given in terms of the delta function by[18]

$$[\mathbf{P}]=\mathbf{P}(\mathbf{r}',t')=\mathbf{p}(t')\delta^3\{\mathbf{r}'-\mathbf{s}(t')\}, \tag{14}$$

$$[\mathbf{M}]=\mathbf{M}(\mathbf{r}',t')=\mathbf{m}(t')\delta^3\{\mathbf{r}'-\mathbf{s}(t')\}, \tag{15}$$

where $\mathbf{p}$ and $\mathbf{m}$ are the electric and magnetic dipole moments and $\mathbf{s}(t')$ is the position of the dipole at the retarded time: $t'=t-|\mathbf{r}-\mathbf{r}'|/c=t-r/c$. The substitution of Eqs. (14) and (15) into Eq. (6) gives the expression

$$\mathbf{E}=-\frac{1}{4\pi}\left\{\frac{\partial}{\partial t}\int\frac{\mathbf{m}(t')\delta^3\{\mathbf{r}'-\mathbf{s}(t')\}\times\hat{r}}{r^2}\,dv' + \frac{\partial^2}{\partial t^2}\int\frac{\mathbf{m}(t')\delta^3\{\mathbf{r}'-\mathbf{s}(t')\}\times\hat{r}}{rc}\,dv'+\cdots\right. . \tag{16}$$

Because of the delta function, the diverse factors of r, $\hat{r}$,..., come outside the integrals where now they are evaluated at the specific retarded position and time defined by $t'=t-|\mathbf{r}-\mathbf{r}'|/c$ and $\mathbf{r}'=\mathbf{s}(t')$. To stress that the vector $\mathbf{r}-\mathbf{r}'$ is now function of the retarded time, we change the notation,

$$\mathbf{r}-\mathbf{r}'\rightarrow\mathbf{R}=R\hat{\mathbf{n}}=\mathbf{r}-\mathbf{s}(t'). \tag{17}$$

Hence, the retarded time $t'=t-r/c$ becomes

$$t'=t-R/c. \tag{18}$$

In the integration of Eq. (16), we use the formula[3]

$$\int\delta^3\{\mathbf{r}'-\mathbf{s}(t')\}dv'=\frac{1}{K}, \tag{19}$$

where

$$K=1-\mathbf{v}\cdot\hat{\mathbf{n}}/c, \tag{20}$$

with $\mathbf{v}$ and $\hat{\mathbf{n}}$ evaluated at the retarded time. After performing all the integrations in Eq. (16), we obtain[19]

$$\mathbf{E}=\frac{1}{4\pi\epsilon_0}\left[\frac{3(\mathbf{p}\cdot\hat{\mathbf{n}})\hat{\mathbf{n}}-\mathbf{p}}{KR^3}+\frac{1}{c}\frac{\partial}{\partial t}\left\{\frac{3(\mathbf{p}\cdot\hat{\mathbf{n}})\hat{\mathbf{n}}-\mathbf{p}}{KR^2}\right\}+\frac{1}{c^2}\frac{\partial^2}{\partial t^2}\left\{\frac{\hat{\mathbf{n}}\times(\hat{\mathbf{n}}\times\mathbf{p})}{KR}\right\}\right]-\frac{1}{4\pi}\left[\frac{\partial}{\partial t}\left\{\frac{\mathbf{m}\times\hat{\mathbf{n}}}{KR^2}\right\}+\frac{1}{c}\frac{\partial^2}{\partial t^2}\left\{\frac{\mathbf{m}\times\hat{\mathbf{n}}}{KR}\right\}\right]. \tag{21}$$

By a similar procedure we find from Eqs. (8), (14), (15), and (19) that the magnetic field is

$$\mathbf{B}=\frac{1}{4\pi}\left[\frac{3(\mathbf{m}\cdot\hat{\mathbf{n}})\hat{\mathbf{n}}-\mathbf{m}}{KR^3}+\frac{1}{c}\frac{\partial}{\partial t}\left\{\frac{3(\mathbf{m}\cdot\hat{\mathbf{n}})\hat{\mathbf{n}}-\mathbf{m}}{KR^2}\right\}+\frac{1}{c^2}\frac{\partial^2}{\partial t^2}\left\{\frac{\hat{\mathbf{n}}\times(\hat{\mathbf{n}}\times\mathbf{m})}{KR}\right\}\right]+\frac{\mu_0}{4\pi}\left[\frac{\partial}{\partial t}\left\{\frac{\mathbf{p}\times\hat{\mathbf{n}}}{KR^2}\right\}+\frac{1}{c}\frac{\partial^2}{\partial t^2}\left\{\frac{\mathbf{p}\times\hat{\mathbf{n}}}{KR}\right\}\right]. \tag{22}$$

The expressions (21) and (22) play the role of intermediate formulas from which we obtain both Heaviside–Feynman and Liénard–Wiechert formulas for a dipole in arbitrary motion. The Heaviside–Feynman form of Eqs. (21) and (22) may be readily obtained. From Eq. (18) we have $\partial t'/\partial t=1-(1/c)\partial R/\partial t$ which combines with $\partial R/\partial t=-(\hat{\mathbf{n}}\cdot\mathbf{v})\partial t'/\partial t$ to give the results

$$\frac{\partial t'}{\partial t}=\frac{1}{1-\hat{\mathbf{n}}\cdot\mathbf{v}/c}=\frac{1}{K}=1-\frac{1}{c}\frac{\partial R}{\partial t}, \tag{23}$$

where we have considered Eq. (20). If we use Eq. (23) in Eqs. (21) and (22) and replace partial derivatives by ordinary derivatives,[20] the result is

$$\mathbf{E}=\frac{1}{4\pi\epsilon_0}\left[\frac{3(\mathbf{p}\cdot\hat{\mathbf{n}})\hat{\mathbf{n}}-\mathbf{p}}{R^3}+\frac{R}{c}\frac{d}{dt}\left(\frac{3(\mathbf{p}\cdot\hat{\mathbf{n}})\hat{\mathbf{n}}-\mathbf{p}}{R^3}\right)-\frac{d}{dt}\left(\frac{3(\mathbf{p}\cdot\hat{\mathbf{n}})\hat{\mathbf{n}}-\mathbf{p}}{R^2c^2}\frac{dR}{dt}+\frac{\epsilon_0\mathbf{m}\times\hat{\mathbf{n}}}{R^2}\left\{1-\frac{1}{c}\frac{dR}{dt}\right\}\right)+\frac{1}{c^2}\frac{d^2}{dt^2}\left(\frac{\hat{\mathbf{n}}\times(\hat{\mathbf{n}}\times\mathbf{p})-c\epsilon_0(\mathbf{m}\times\hat{\mathbf{n}})}{R}\left\{1-\frac{1}{c}\frac{dR}{dt}\right\}\right)\right], \tag{24}$$

$$\mathbf{B}=\frac{1}{4\pi}\left[\frac{3(\mathbf{m}\cdot\hat{\mathbf{n}})\hat{\mathbf{n}}-\mathbf{m}}{R^3}+\frac{R}{c}\frac{d}{dt}\left(\frac{3(\mathbf{m}\cdot\hat{\mathbf{n}})\hat{\mathbf{n}}-\mathbf{m}}{R^3}\right)-\frac{d}{dt}\left(\frac{3(\mathbf{m}\cdot\hat{\mathbf{n}})\hat{\mathbf{n}}-\mathbf{m}}{R^2c^2}\frac{dR}{dt}-\frac{\mu_0\mathbf{P}\times\hat{\mathbf{n}}}{R^2}\left\{1-\frac{1}{c}\frac{dR}{dt}\right\}\right)+\frac{1}{c^2}\frac{d^2}{dt^2}\left(\frac{\hat{\mathbf{n}}\times(\hat{\mathbf{n}}\times\mathbf{m})+c\mu_0(\mathbf{p}\times\hat{\mathbf{n}})}{R}\times\left\{1-\frac{1}{c}\frac{dR}{dt}\right\}\right)\right]. \tag{25}$$

These are the Heaviside–Feynman formulas for a dipole in arbitrary motion. Monaghan[8] derived these formulas using the standard method of potentials. The interpretation of the terms in Eqs. (24) and (25) does not appear to be simple. The first term in both expressions gives the static field of the dipole at its retarded position while the second term gives the first-order correction for this static field; we have no a simple interpretation for the third term; and finally the last term contains, at least in principle, all the radiation effects. In order to interpret in more familiar terms the fields of a dipole in arbitrary motion we should like to put them in the Liénard–Wiechert form.

IV. THE RADIATION FIELDS OF A DIPOLE IN ARBITRARY MOTION

We have already mentioned that the Liénard–Wiechert form for the fields of a dipole in arbitrary motion may be derived from the intermediate formulas given in Eqs. (21) and (22). Such a derivation, though straightforward, is extremely long and the full expressions obtained for the fields turn out to be very lengthy. Accordingly, the complete derivation will be not presented here. However, we can say that these fields may be separated naturally in terms of the form $O(1/R)$, $O(1/R^2)$, and $O(1/R^3)$. Moreover, these terms involve complicated combinations of $\mathbf{p}$, $\dot{\mathbf{p}}$, $\ddot{\mathbf{p}}$, $\mathbf{m}$, $\dot{\mathbf{m}}$, $\ddot{\mathbf{m}}$, $\mathbf{v}$, $\mathbf{a}$, and $\dot{\mathbf{a}}$ (!)—this fact throws into relief the beautiful simplicity of the Heaviside–Feynman form given in Eqs. (24) and (25). We shall be content with the much less ambitious task of finding those terms of the Liénard–Wiechert form which are relevant for the radiation effects, that is, those terms that give the radiation field at great distance from the dipole. These terms vary like $1/R$ and their deduction is not very difficult. Let us begin by stating the derivatives

$$\frac{\partial \mathbf{R}}{\partial t}=\frac{\partial}{\partial t}\{\mathbf{r}-\mathbf{s}(t')\}=-\frac{\partial \mathbf{s}(t')}{\partial t}=\frac{\partial \mathbf{s}(t')}{\partial t'}\frac{\partial t'}{\partial t}=-\frac{\mathbf{v}}{K}, \tag{26}$$

$$\frac{\partial R}{\partial t}=\frac{\partial}{\partial t}(\mathbf{R}\cdot\mathbf{R})^{1/2}=\hat{\mathbf{n}}\cdot\frac{\partial \mathbf{R}}{\partial t}=-\frac{\hat{\mathbf{n}}\cdot\mathbf{v}}{K}, \tag{27}$$

$$\frac{\partial \hat{\mathbf{n}}}{\partial t}=\frac{\partial}{\partial t}\left(\frac{\mathbf{R}}{R}\right)=\frac{1}{RK}\{(\hat{\mathbf{n}}\cdot\mathbf{v})\hat{\mathbf{n}}-\mathbf{v}\}=\frac{\hat{\mathbf{n}}\times(\hat{\mathbf{n}}\times\mathbf{v})}{RK}, \tag{28}$$

$$\frac{\partial K}{\partial t}=\frac{\partial}{\partial t}(1-\mathbf{v}\cdot\hat{\mathbf{n}}/c)=-\frac{\partial}{\partial t}\left(\frac{\mathbf{v}\cdot\hat{\mathbf{n}}}{c}\right)=-\frac{1}{K}\left\{\frac{\hat{\mathbf{n}}\cdot\mathbf{a}}{c}+\frac{(\hat{\mathbf{n}}\cdot\mathbf{v})^2}{Rc}-\frac{\mathbf{v}\cdot\mathbf{v}}{Rc}\right\}, \tag{29}$$

$$\frac{\partial}{\partial t}(R^iK^j)=jR^iK^{j-2}\left\{-\frac{\hat{\mathbf{n}}\cdot\mathbf{a}}{c}-\frac{(\hat{\mathbf{n}}\cdot\mathbf{v})^2}{Rc}+\frac{\mathbf{v}\cdot\mathbf{v}}{Rc}\right\}-iR^{i-1}K^{j-1}(\hat{\mathbf{n}}\cdot\mathbf{v}),\quad (i,j\ \text{integers}). \tag{30}$$

Now, If we expand the following term of Eq. (21)

$$\frac{1}{c^2}\frac{\partial^2}{\partial t^2}\left\{\frac{\hat{\mathbf{n}}\times(\hat{\mathbf{n}}\times\mathbf{p})}{RK}\right\},$$

we can rewrite Eq. (21) itself in a more manageable form

$$\mathbf{E}=\frac{1}{4\pi\epsilon_0}\left[\frac{3(\mathbf{p}\cdot\hat{\mathbf{n}})\hat{\mathbf{n}}-\mathbf{p}}{R^3K}+\frac{1}{c}\frac{\partial}{\partial t}\left\{\frac{3(\mathbf{p}\cdot\hat{\mathbf{n}})\hat{\mathbf{n}}-\mathbf{p}}{R^2K}\right\}+\left(\frac{\mathbf{p}\times\hat{\mathbf{n}}}{RK}\right)\times\frac{1}{c^2}\frac{\partial^2\hat{\mathbf{n}}}{\partial t^2}+\frac{1}{c}\frac{\partial}{\partial t}\left(\frac{\mathbf{p}\times\hat{\mathbf{n}}}{RK}\right)\times\frac{2}{c}\frac{\partial\hat{\mathbf{n}}}{\partial t}+\frac{1}{c}\frac{\partial^2}{\partial t^2}\left(\frac{\mathbf{p}\times\hat{\mathbf{n}}}{RK}\right)\times\frac{\hat{\mathbf{n}}}{c}\right]-\frac{1}{4\pi}\left[\frac{\partial}{\partial t}\left(\frac{\mathbf{m}\times\hat{\mathbf{n}}}{R^2K}\right)+\frac{1}{c}\frac{\partial^2}{\partial t^2}\left(\frac{\mathbf{m}\times\hat{\mathbf{n}}}{RK}\right)\right]. \tag{31}$$

Let us label the quantities inside the square brackets of Eq. (32) as follows:

$$[1]=\frac{3(\mathbf{p}\cdot\hat{\mathbf{n}})\hat{\mathbf{n}}-\mathbf{p}}{R^3K}+\frac{1}{c}\frac{\partial}{\partial t}\left\{\frac{3(\mathbf{p}\cdot\hat{\mathbf{n}})\hat{\mathbf{n}}-\mathbf{p}}{R^2K}\right\},$$

$$[2]=\left(\frac{\mathbf{p}\times\hat{\mathbf{n}}}{RK}\right)\times\frac{1}{c^2}\frac{\partial^2\hat{\mathbf{n}}}{\partial t^2},$$

$$[3]=\frac{1}{c}\frac{\partial}{\partial t}\left(\frac{\mathbf{p}\times\hat{\mathbf{n}}}{RK}\right)\times\frac{2}{c}\frac{\partial\hat{\mathbf{n}}}{\partial t},$$

$$[4]=\frac{1}{c}\frac{\partial^2}{\partial t^2}\left(\frac{\mathbf{p}\times\hat{\mathbf{n}}}{RK}\right)\times\frac{\hat{\mathbf{n}}}{c},$$

$$[5]=\frac{\partial}{\partial t}\left(\frac{\mathbf{m}\times\hat{\mathbf{n}}}{R^2K}\right)+\frac{1}{c}\frac{\partial^2}{\partial t^2}\left(\frac{\mathbf{m}\times\hat{\mathbf{n}}}{RK}\right).$$

Bearing in mind Eqs. (26)–(30) we shall examine each of the quantities [1]–[5]. Specifically our work will consist in extracting the terms varying like $1/R$ from these quantities. It is not difficult to prove that the quantity [1] contains exclusively terms of the form $O(1/R^2)$ and $O(1/R^3)$. Hence [1] does not contribute to radiation. The time derivative of Eq. (28) shows that the factor $\partial^2\hat{\mathbf{n}}/\partial t^2$ inside [2] contains only terms of the form $O(1/R)$ and $O(1/R^2)$. Hence the quantity [2] produces only terms of the form $O(1/R^2)$ and $O(1/R^3)$. The quantity [3] is more difficult to analyze. With a bit of manipulation we find the following expression for the first factor inside [3]:

$$\frac{1}{c}\frac{\partial}{\partial t}\left(\frac{\mathbf{p}\times\hat{\mathbf{n}}}{RK}\right)=\frac{\mathbf{p}\times\hat{\mathbf{n}}}{\gamma^2R^2K^3}-\frac{\mathbf{p}\times\hat{\mathbf{n}}}{R^2K}-\frac{\mathbf{p}\times\mathbf{v}}{R^2K^2c}+\frac{\dot{\mathbf{p}}\times\hat{\mathbf{n}}}{RK^2c}+\frac{(\mathbf{p}\times\hat{\mathbf{n}})(\hat{\mathbf{n}}\cdot\mathbf{a})}{RK^3c^2}, \tag{32}$$

where as usual $\gamma=(1-v^2/c^2)^{-1/2}$. Evidently, this factor contains terms of order $O(1/R)$ and $O(1/R^2)$ exclusively. Therefore, since $\partial\hat{\mathbf{n}}/\partial t$ carries a factor of order $O(1/R)$, the quantity [3] contributes only terms of the form $O(1/R^2)$ and $O(1/R^3)$.

In the examination of [4] we need first the time derivative of Eq. (32),

$$\frac{1}{c}\frac{\partial^2}{\partial t^2}\left(\frac{\mathbf{p}\times\hat{\mathbf{n}}}{RK}\right)=\frac{\partial}{\partial t}\left(\frac{\mathbf{p}\times\hat{\mathbf{n}}}{\gamma^2R^2K^3}\right)-\frac{\partial}{\partial t}\left(\frac{\mathbf{p}\times\hat{\mathbf{n}}}{R^2K}\right)-\frac{\partial}{\partial t}\left(\frac{\mathbf{p}\times\mathbf{v}}{R^2K^2c}\right)+\frac{\partial}{\partial t}\left(\frac{\dot{\mathbf{p}}\times\hat{\mathbf{n}}}{RK^2c}\right)+\frac{\partial}{\partial t}\left\{\frac{(\mathbf{p}\times\hat{\mathbf{n}})(\hat{\mathbf{n}}\cdot\mathbf{a})}{RK^3c^2}\right\}. \tag{33}$$

This is the first factor in [4]. It is evident that the first three quantities of the right-hand side of Eq. (33) contain only terms of the form $O(1/R^2)$ and $O(1/R^3)$. However, the last two quantities of Eq. (33) contain terms of the form $O(1/R)$. These terms are

$$\frac{\ddot{\mathbf{p}}\times\hat{\mathbf{n}}}{RK^3c}+\frac{2(\dot{\mathbf{p}}\times\hat{\mathbf{n}})(\hat{\mathbf{n}}\cdot\mathbf{a})}{RK^4c^2},$$

$$\frac{(\dot{\mathbf{p}}\times\hat{\mathbf{n}})(\hat{\mathbf{n}}\cdot\mathbf{a})}{RK^4c^2}+\frac{3(\mathbf{p}\times\hat{\mathbf{n}})(\hat{\mathbf{n}}\cdot\mathbf{a})^2}{RK^5c^3}+\frac{(\mathbf{p}\times\hat{\mathbf{n}})(\hat{\mathbf{n}}\cdot\dot{\mathbf{a}})}{RK^4c^2}.$$

In the last term there appears the surprising factor $\dot{\mathbf{a}}$, i.e., the time derivative of the acceleration! We conclude that the terms $O(1/R)$ of the first factor inside [4] are

$$\frac{3(\mathbf{p}\times\hat{\mathbf{n}})(\hat{\mathbf{n}}\cdot\mathbf{a})^2}{RK^5c^3}+\frac{3(\dot{\mathbf{p}}\times\hat{\mathbf{n}})(\hat{\mathbf{n}}\cdot\mathbf{a})}{RK^4c^2}+\frac{(\mathbf{p}\times\hat{\mathbf{n}})(\hat{\mathbf{n}}\cdot\dot{\mathbf{a}})}{RK^4c^2}+\frac{\ddot{\mathbf{p}}\times\hat{\mathbf{n}}}{RK^3c}. \tag{34}$$

With this result we conclude that the terms $O(1/R)$ of [4] are

$$\frac{3\hat{\mathbf{n}}\times(\hat{\mathbf{n}}\times\mathbf{p})(\hat{\mathbf{n}}\cdot\mathbf{a})^2}{RK^5c^4}+\frac{3\hat{\mathbf{n}}\times(\hat{\mathbf{n}}\times\dot{\mathbf{p}})(\hat{\mathbf{n}}\cdot\mathbf{a})}{RK^4c^3}+\frac{\hat{\mathbf{n}}\times(\hat{\mathbf{n}}\times\mathbf{p})(\hat{\mathbf{n}}\cdot\dot{\mathbf{a}})}{RK^4c^3}+\frac{\hat{\mathbf{n}}\times(\hat{\mathbf{n}}\times\ddot{\mathbf{p}})}{RK^3c^2}. \tag{35}$$

There remains the quantity [5]. The first element on the right-hand side of [5] is of the form $O(1/R^2)$ whereas the second element has the same form of the first factor inside [4] but with m instead of $\mathbf{p}$. Consequently we can easily infer that the terms $O(1/R)$ of [5] are given by the expression (34) but with $\mathbf{m}$ instead of $\mathbf{p}$, i.e., by

$$\frac{3(\mathbf{m}\times\hat{\mathbf{n}})(\hat{\mathbf{n}}\cdot\mathbf{a})^2}{RK^5c^3}+\frac{3(\dot{\mathbf{m}}\times\hat{\mathbf{n}})(\hat{\mathbf{n}}\cdot\mathbf{a})}{RK^4c^2}+\frac{(\mathbf{m}\times\hat{\mathbf{n}})(\hat{\mathbf{n}}\cdot\dot{\mathbf{a}})}{RK^4c^2}+\frac{\ddot{\mathbf{m}}\times\hat{\mathbf{n}}}{RK^3c}. \tag{36}$$

Returning now to the expression (31) for the electric field, we see that its radiation terms are given by the terms of the form $O(1/R)$ of the quantities [4] and [5]. Hence the results in Eqs. (35) and (36) give us the radiation field "$\mathbf{E}_{\rm rad}$" for a dipole in arbitrary motion

$$\mathbf{E}_{\rm rad}=\frac{1}{4\pi\epsilon_0}\left[\frac{3\hat{\mathbf{n}}\times(\hat{\mathbf{n}}\times\mathbf{p}+\epsilon_0c\mathbf{m})(\hat{\mathbf{n}}\cdot\mathbf{a})^2}{RK^5c^4}+\frac{3\hat{\mathbf{n}}\times(\hat{\mathbf{n}}\times\dot{\mathbf{p}}+\epsilon_0c\dot{\mathbf{m}})(\hat{\mathbf{n}}\cdot\mathbf{a})}{RK^4c^3}+\frac{\hat{\mathbf{n}}\times(\hat{\mathbf{n}}\times\mathbf{p}+\epsilon_0c\mathbf{m})(\hat{\mathbf{n}}\cdot\dot{\mathbf{a}})}{RK^4c^3}+\frac{\hat{\mathbf{n}}\times(\hat{\mathbf{n}}\times\ddot{\mathbf{p}}+\epsilon_0c\ddot{\mathbf{m}})}{RK^3c^2}\right]. \tag{37}$$

Proceeding in a similar way, we find the formula for the radiation field "$\mathbf{B}_{\rm rad}$" for a dipole in arbitrary motion

$$\mathbf{B}_{\rm rad}=\frac{1}{4\pi}\left[\frac{3\hat{\mathbf{n}}\times(\hat{\mathbf{n}}\times\mathbf{m}-\mu_0c\mathbf{p})(\hat{\mathbf{n}}\cdot\mathbf{a})^2}{RK^5c^4}+\frac{3\hat{\mathbf{n}}\times(\hat{\mathbf{n}}\times\dot{\mathbf{m}}-\mu_0c\dot{\mathbf{p}})(\hat{\mathbf{n}}\cdot\mathbf{a})}{RK^4c^3}+\frac{\hat{\mathbf{n}}\times(\hat{\mathbf{n}}\times\mathbf{m}-\mu_0c\mathbf{p})(\hat{\mathbf{n}}\cdot\dot{\mathbf{a}})}{RK^4c^3}+\frac{\hat{\mathbf{n}}\times(\hat{\mathbf{n}}\times\ddot{\mathbf{m}}-\mu_0c\ddot{\mathbf{p}})}{RK^3c^2}\right]. \tag{38}$$

The formulas (37) and (38) constitute our version of the radiation fields of an arbitrarily moving dipole having both electric and magnetic moments.[21] In other set of words: Eqs. (37) and (38) represent effectively the radiative part of the Liénard–Wiechert fields of a dipole in arbitrary motion. In particular if we consider the moments $\mathbf{p}$ and $\mathbf{m}$ as defined by Eqs. (9) and (10), then the formulas (37) and (38) give us the radiative part of the fields defined in Eqs. (12) and (13). Apart from their manifest conciseness and symmetry, Eqs. (37) and (38) are particularly illuminating because each term can be interpreted without great difficulty.

Let us consider, for example, the terms of Eq. (37)—the interpretation of the terms in Eq. (38) is completely analogous. The first term is a novel one: It survives even when $\mathbf{a}$, $\mathbf{p}$, and $\mathbf{m}$ are all constant quantities. However, the radiation produced by a dipole of *time-constant moment* $\mathbf{p}$ or $\mathbf{m}$, but accelerated in *position*, does not appear to be a *familiar* or *expected* result. The familiar idea is that accelerated *charges* produce radiation, but a dipole is a configuration with net charge equal to zero—as Griffiths[14] has written recently: "And yet, the theory of dipoles contains a rich measure of subtlety and surprise." The second term is one more familiar. It arises from the time dependence of $\mathbf{p}$ and $\mathbf{m}$ as well as the acceleration of the dipole. The third term is also novel one since it contains the *unusual* time derivative of the acceleration. This is a really unexpected result. Since the time derivative of the acceleration can be nonzero even when the acceleration itself is instantaneously zero, it follows that *an instantaneously unaccelerated dipole of time-constant moment* $\mathbf{p}$ *or* $\mathbf{m}$ *produces radiation provided the time-derivative of the acceleration is nonzero*! The fourth term (which is independent of the acceleration) is one already known. We find it in fields of stationary configurations with time-dependent dipole moments.[22] In virtue of this term, if the dipole is moving with constant velocity, or even more, when it is at rest we find that it produces radiation if $\ddot{\mathbf{p}}$ and/or $\ddot{\mathbf{m}}$ is nonzero.

Finally notice that Eq. (38) can be written as

$$\mathbf{B}_{\rm rad}=(1/c)[\hat{\mathbf{n}}]\times\mathbf{E}_{\rm rad}, \tag{39}$$

as expected. Consequently $\mathbf{B}_{\rm rad}$ and $\mathbf{E}_{\rm rad}$ are perpendicular to one another, i.e., they satisfy $\mathbf{B}_{\rm rad}\cdot\mathbf{E}_{\rm rad}=0$. It should be noted also that Eq. (37) can be written as $\mathbf{E}_{\rm rad}=[\hat{\mathbf{n}}]\times\{\cdots\}$. Hence we have $\mathbf{E}_{\rm rad}\cdot[\hat{\mathbf{n}}]=0$ and $\mathbf{B}_{\rm rad}\cdot[\hat{\mathbf{n}}]=0$, i.e., both $\mathbf{E}_{\rm rad}$ and $\mathbf{B}_{\rm rad}$ fields are perpendicular to the vector $[\hat{\mathbf{n}}]$.

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APPENDIX: DERIVATION OF EQS. (4), (5), AND (7)

The proof of Eq. (4) is as follows. If we integrate the equation

$$\frac{[\nabla'\times\mathbf{M}]}{r}=\nabla\times\frac{[\mathbf{M}]}{r}+\nabla'\times\frac{[\mathbf{M}]}{r}, \tag{A1}$$

over all space we have

$$\int\frac{[\nabla'\times\mathbf{M}]}{r}\,dv'=\int\nabla\times\frac{[\mathbf{M}]}{r}\,dv'+\int\nabla'\times\frac{[\mathbf{M}]}{r}\,dv'. \tag{A2}$$

The second integral on the right-hand side can be transformed into a surface integral

$$\int\nabla'\times\frac{[\mathbf{M}]}{r}\,dv'=-\oint\frac{[\mathbf{M}]}{r}\times d\mathbf{S}'. \tag{A3}$$

But since $\mathbf{M}$ is zero outside a finite region of space ($\mathbf{M}$ is confined in this region), while the surface integral in Eq. (A3) is taken over all space, this last integral vanishes. Therefore Eq. (A2) reduces to

$$\int \frac{[\nabla'\times\mathbf{M}]}{r}\,dv' = \int \nabla\times\frac{[\mathbf{M}]}{r}\,dv'$$
$$= \int \left\{ \frac{1}{r}\nabla\times[\mathbf{M}] - [\mathbf{M}]\times\nabla\left(\frac{1}{r}\right) \right\} dv'. \tag{A4}$$

Following Jefimenko, we note that $[\mathbf{M}]$ depends implicitly on the field point $\mathbf{r}$ through the retarded time, and hence we can use the *chain rule* to obtain

$$\nabla\times[\mathbf{M}] = \frac{1}{c}\frac{\partial}{\partial t}[\mathbf{M}]\times\hat{r}. \tag{A5}$$

Using this equation along with $\nabla(1/r) = -\hat{r}/r^2$ into Eq. (A4) we obtain

$$\int \frac{[\nabla'\times\mathbf{M}]}{r}\,dv' = \int \left\{ \frac{[\mathbf{M}]\times\hat{r}}{r^2} + \frac{(\partial[\mathbf{M}]/\partial t)\times\hat{r}}{rc} \right\} dv'. \tag{A6}$$

Accordingly, the time derivative of Eq. (A6) gives directly Eq. (4).

The proof of Eq. (5) turns out to be more elaborated. Consider first the relation

$$\nabla\int \frac{[\nabla'\cdot\mathbf{P}]}{r}\,dv' = -\int \left\{ \frac{[\nabla'\cdot\mathbf{P}]\hat{r}}{r^2} + \frac{(\partial[\nabla'\cdot\mathbf{P}]/\partial t)\hat{r}}{rc} \right\} dv'. \tag{A7}$$

The demonstration of this is as follows:

$$\nabla\int \frac{[\nabla'\cdot\mathbf{P}]}{r}\,dv' = \int \nabla\left\{\frac{[\nabla'\cdot\mathbf{P}]}{r}\right\} dv'$$
$$= \int \left\{ \frac{1}{r}\nabla[\nabla'\cdot\mathbf{P}] + [\nabla'\cdot\mathbf{P}]\nabla\left(\frac{1}{r}\right) \right\} dv'. \tag{A8}$$

Now, we note that $[\nabla'\cdot\mathbf{P}]$ depends implicitly on the field point $\mathbf{r}$ through the retarded time. Therefore, using Jefimenko's trick we obtain

$$\nabla[\nabla'\cdot\mathbf{P}] = -\frac{1}{c}\left\{ \frac{\partial}{\partial t}[\nabla'\cdot\mathbf{P}] \right\}\hat{r}. \tag{A9}$$

The use of Eq. (A9) along with $\nabla(1/r) = -\hat{r}/r^2$ in Eq. (A8) give us Eq. (A7).

Our second step is to rewrite the left-hand side of Eq. (A7). The integration of the equation

$$\frac{[\nabla'\cdot\mathbf{P}]}{r} = \nabla\cdot\frac{[\mathbf{P}]}{r} + \nabla'\cdot\frac{[\mathbf{P}]}{r}, \tag{A10}$$

over all space gives

$$\int \frac{[\nabla'\cdot\mathbf{P}]}{r}\,dv' = \int \nabla\cdot\frac{[\mathbf{P}]}{r}\,dv' + \int \nabla'\cdot\frac{[\mathbf{P}]}{r}\,dv'. \tag{A11}$$

The second integral on the right-hand side can be transformed into a surface integral

$$\int \nabla'\cdot\frac{[\mathbf{P}]}{r}\,dv' = \oint \frac{[\mathbf{P}]}{r}\cdot d\mathbf{S}'. \tag{A12}$$

But since $\mathbf{P}$ is confined in a finite region of space, while the surface integral in Eq. (A12) is taken over all space, this last integral vanishes. Hence, Eq. (A11) reduces to

$$\int \frac{[\nabla'\cdot\mathbf{P}]}{r}\,dv' = \int \nabla\cdot\frac{[\mathbf{P}]}{r}\,dv'$$
$$= \int \left\{ \frac{1}{r}\nabla\cdot[\mathbf{P}] + [\mathbf{P}]\cdot\nabla\left(\frac{1}{r}\right) \right\} dv'. \tag{A13}$$

The vector $[\mathbf{P}]$ depends implicitly on $\mathbf{r}$ through the retarded time. Hence,

$$\nabla\cdot[\mathbf{P}] = -\frac{1}{c}[\partial\mathbf{P}/\partial t]\cdot\hat{r}. \tag{A14}$$

The use of this equation along with $\nabla(1/r) = -\hat{r}/r^2$ in Eq. (A11) give us

$$\int \frac{[\nabla'\cdot\mathbf{P}]}{r}\,dv' = -\int \left\{ \frac{[\mathbf{P}]\cdot\hat{r}}{r^2} + \frac{[\partial\mathbf{P}/\partial t]\cdot\hat{r}}{rc} \right\} dv'. \tag{A15}$$

Now, the gradient of Eq. (A15) is

$$\nabla\int \frac{[\nabla'\cdot\mathbf{P}]}{r}\,dv' = -\int \left(\nabla\left\{\frac{[\mathbf{P}]\cdot\hat{r}}{r^2}\right\} + \nabla\left\{\frac{[\partial\mathbf{P}/\partial t]\cdot\hat{r}}{rc}\right\} \right) dv'. \tag{A16}$$

Our next step is to evaluate the gradients inside the integral sign on the right-hand side of Eq. (A16). After a laborious calculation (in which we use the vector identity: $\nabla(\mathbf{A}\cdot\mathbf{B}) = \mathbf{A}\times(\nabla\times\mathbf{B}) + \mathbf{B}\times(\nabla\times\mathbf{A}) + (\mathbf{A}\cdot\nabla)\mathbf{B} + (\mathbf{B}\cdot\nabla)\mathbf{A}$, as well as Jefimenko's trick that spatial derivatives of retarded quantities may be converted into time derivatives), we obtain the following equations:

$$\nabla\left\{\frac{[\mathbf{P}]\cdot\hat{r}}{r^2}\right\} = \frac{[\mathbf{P}] - 3([\mathbf{P}]\cdot\hat{r})\hat{r}}{r^3} - \frac{([\partial\mathbf{P}/\partial t]\cdot\hat{r})\hat{r}}{r^2 c}, \tag{A17}$$

$$\nabla\left\{\frac{[\partial\mathbf{P}/\partial t]\cdot\hat{r}}{rc}\right\} = \frac{[\partial\mathbf{P}/\partial t] - 2([\partial\mathbf{P}/\partial t]\cdot\hat{r})\hat{r}}{r^2 c} - \frac{([\partial^2\mathbf{P}/\partial t^2]\cdot\hat{r})\hat{r}}{rc^2}. \tag{A18}$$

The substitution of Eqs. (A17) and (A18) into Eq. (A16) produces the result

$$\nabla\int \frac{[\nabla'\cdot\mathbf{P}]}{r}\,dv' = \int \left\{ \frac{3([\mathbf{P}]\cdot\hat{r})\hat{r} - [\mathbf{P}]}{r^3} + \frac{3([\partial\mathbf{P}/\partial t]\cdot\hat{r})\hat{r} - [\partial\mathbf{P}/\partial t]}{r^2 c} + \frac{([\partial^2\mathbf{P}/\partial t^2]\cdot\hat{r})\hat{r}}{rc^2} \right\} dv'. \tag{A19}$$

From Eqs. (A7) and (A19) we deduce directly Eq. (5). By a similar procedure we shall establish Eq. (7). Consider first the relation

$$\nabla\times\int \frac{[\nabla'\times\mathbf{M}]}{r}\,dv' = \int \left\{ \frac{[\nabla'\times\mathbf{M}]\times\hat{r}}{r^2} + \frac{(\partial[\nabla'\times\mathbf{M}]/\partial t)\times\hat{r}}{rc} \right\} dv'. \tag{A20}$$

The proof of this is as follows:

$$\nabla\times\int\frac{[\nabla'\times\mathbf{M}]}{r}\,dv'=\int\nabla\times\left\{\frac{[\nabla'\times\mathbf{M}]}{r}\right\}dv'$$
$$=\int\left\{\frac{1}{r}\nabla\times[\nabla'\times\mathbf{M}]-[\nabla'\times\mathbf{M}]\times\nabla\left(\frac{1}{r}\right)\right\}dv'. \qquad \text{(A21)}$$

As before $[\nabla'\times\mathbf{M}]$ depends implicitly on $\mathbf{r}$ through the retarded time. Thus

$$\nabla\times[\nabla'\times\mathbf{M}]=\frac{1}{c}\left\{\frac{\partial}{\partial t}[\nabla'\times\mathbf{M}]\right\}\times\hat{r}. \qquad \text{(A22)}$$

Using Eq. (A22) into Eq. (A21) we obtain Eq. (A20). We need now to rewrite the left-hand side of Eq. (A20). The curl of Eq. (A6) along with the equations

$$\nabla\times\left\{\frac{[\mathbf{M}]\times\hat{r}}{r^2}\right\}=\frac{3([\mathbf{M}]\cdot\hat{r})\hat{r}-[\mathbf{M}]}{r^3}+\frac{([\partial\mathbf{M}/\partial t]\cdot\hat{r})\hat{r}-[\partial\mathbf{M}/\partial t]}{r^2c}, \qquad \text{(A23)}$$

$$\nabla\times\left\{\frac{[\partial\mathbf{M}/\partial t]\times\hat{r}}{rc}\right\}=\frac{2([\partial\mathbf{M}/\partial t]\cdot\hat{r})\hat{r}}{r^2c}+\frac{([\partial^2\mathbf{M}/\partial t^2]\cdot\hat{r})\hat{r}-[\partial^2\mathbf{M}/\partial t^2]}{rc^2}, \qquad \text{(A24)}$$

imply directly the expression

$$\nabla\times\int\frac{[\nabla'\times\mathbf{M}]}{r}\,dv'=\int\left\{\frac{[\mathbf{M}]-3([\mathbf{M}]\cdot\hat{r})\hat{r}}{r^3}+\frac{[\partial\mathbf{M}/\partial t]-3([\partial\mathbf{M}/\partial t]\cdot\hat{r})\hat{r}}{r^2c}+\frac{[\partial^2\mathbf{M}/\partial t^2]-([\partial^2\mathbf{M}/\partial t^2]\cdot\hat{r})\hat{r}}{rc^2}\right\}dv'. \qquad \text{(A25)}$$

In the determination of Eqs. (A23) and (A24), we have used the vector identity: $\nabla\times(\mathbf{A}\times\mathbf{B})=\mathbf{A}(\nabla\cdot\mathbf{B})-\mathbf{B}(\nabla\cdot\mathbf{A})+(\mathbf{B}\cdot\nabla)\mathbf{A}-(\mathbf{A}\cdot\nabla)\mathbf{B}$ as well as Jefimenko's trick. From Eqs. (A20) and (A25) we infer Eq. (7) and the proof ends.

[1]Oleg D. Jefimenko, *Electricity and Magnetism*, 2nd ed. (Electrect Scientific, Star City, 1989), p 516; see also the recent book of this author, *Causality, Electromagnetic Induction and Gravitation* (Electrect Scientific, Star City, 1992), Sec. 1.4.

[2]Oleg D. Jefimenko, "Comment on 'On the equivalence of the laws of Biot-Savart and Ampere,' by Weber and Macomb," Am. J Phys. **58**, 505 (1990).

[3]David J. Griffiths and M. A. Heald, "Time-dependent generalizations of the Biot–Savart and Coulomb Laws," Am. J. Phys. **59**, 111–117 (1991).

[4]Tran-Cong Ton, "On the time-dependent, generalized Coulomb and Biot-Savart Laws," Am. J. Phys. **59**, 520–528 (1991).

[5]Oleg D. Jefimenko, "Force exerted on a stationary charge by a moving electric current or by a moving magnet," Am. J. Phys. **61**, 218–222 (1993).

[6]José A. Heras, "Jefimenko's formulas with magnetic monopoles and the Liénard-Wiechert fields of a dual-charged particle," Am. J. Phys. **62**, 525–531 (1994).

[7]Oleg D. Jefimenko, "Solutions of Maxwell's equations for electric and magnetic fields in arbitrary media," Am. J. Phys. **60**, 899–902 (1992).

[8]J. J. Monaghan, "The Heaviside–Feynman expression for the fields of an accelerated dipole," J. Phys. A **1**, 112–117 (1968).

[9]J. R. Ellis, "The fields of an arbitrarily moving dipole," Proc. Cambridge Philos. Soc. **59**, 759–774 (1963).

[10]G. N. Ward, "On the integration of Maxwell's equations, and charge and dipole fields," Proc. R. Soc. London, Ser. A **279**, 562–571 (1964).

[11]G. N. Ward, "The electromagnetic fields of moving dipoles," Proc. Cambridge Philos. Soc. **61**, 547–553 (1965)

[12]J. R. Ellis, "Electromagnetic fields of moving dipoles and multipoles," J. Math. Phys. **7**, 1185–1197 (1966).

[13]The formulas (1) and (2) deserve some comments: (i) In his original formulation, Jefimenko wrote Eqs. (1) and (2) in terms of the "fictitious" polarization and magnetization charge densities, $\rho_P=-\nabla\cdot\mathbf{P}$ and $\rho_M=-\nabla\cdot\mathbf{M}$, as well as in terms of the "fictitious" polarization and magnetization current densities, $\mathbf{J}_P=(1/\epsilon_0)\nabla\times\mathbf{P}$ and $\mathbf{J}_M=(1/\epsilon_0)\nabla\times\mathbf{M}$. In this paper we prefer to avoid the use of the symbols ρ_P, ρ_M, $\mathbf{J}_P$, and $\mathbf{J}_M$ since we want all partial derivatives to appear explicitly. Moreover, Jefimenko's unconventional use of the symbol $\mathbf{J}_P$ to designate the so-called fictitious polarization current leads to confusion since this symbol is commonly used to designate the polarization current $\partial\mathbf{P}/\partial t$; (ii) The operations involving terms of the form $[\nabla'\cdot\mathbf{P}]$, $[\nabla'\times\mathbf{M}]$,... inside of Eqs. (1) and (2), must be interpreted unambiguously. For example, $[\nabla'\cdot\mathbf{P}]$ means that $\nabla'\cdot\mathbf{P}(\mathbf{r}',t)$ first, then substitute in retarded time for t. That is, the ∇'s does not act on the $\mathbf{r}'$ in the retarded time t'; and finally, (iii) We do not consider here the formulas for the auxiliary fields $\mathbf{D}$ and $\mathbf{H}$ derived by Jefimenko [Eqs. (31) and (32) of Ref. 7]. These formulas for $\mathbf{D}$ and $\mathbf{H}$ are actually unnecessary. The vector $\mathbf{D}$, for example, is defined by $\mathbf{D}=\epsilon_0\mathbf{E}+\mathbf{P}$, and $\mathbf{P}$ is a given function in our context. A similar argument applies to the field $\mathbf{H}$. It should be noted that in practical situations where the retardation is important, the field point is almost always in free-space (even although the sources may include time-varying $\mathbf{P}$ and $\mathbf{M}$, as in the applications given in this paper).

[14]For a review of dipoles at rest see, for example, the recent article of David J. Griffiths, "Dipoles at rest," Am. J. Phys. **60**, 979–987 (1992).

[15]See, e.g., Jerry B. Marion and Mark A. Heald, *Classical Electromagnetic Radiation* 2nd ed. (Academic, New York, 1980), Sec. 8.4 and problem 8-3; J. R. Reitz, F. J. Milford, and R. W. Christy, *Foundations of Electromagnetic Theory*, 4th ed. (Addison–Wesley, Reading, MA, 1993), Sec. 20.4.

[16]See, for instance, J. B. Marion and M. A. Heald in Ref. 15, Sec. 8.10.

[17]The present author has derived the full Liénard–Wiechert fields of a dipole in arbitrary motion in terms of conventional parameters (the total fields include near, intermediate and far contributions). However, the structure of these fields is extremely long, so that they deserve a detailed discussion which will be presented elsewhere. In passing, it is interesting to point out that, with some restrictions, Ellis in Ref. 9 has written the fields of the dipole in a form close to the Liénard–Wiechert form derived by the present author. However, Ellis' treatment involves unfamiliar parameters which make difficult the understanding of the fields. The comparison of Ellis' fields with those derived by the present author does not appear to be a simple task.

[18]In a sense Eqs. (14) and (15) beg the essential question: What are the dipole moments? We know that a moving electric dipole acquires a magnetic dipole moment (and vice versa), but precisely *how*—and whether the answer is model dependent—has been the subject of controversy and debate. This question never arises in the case of point charge, because the point charge is *conserved* and *invariant*, so it is necessarily independent of the motion (and hence of t'). But $\mathbf{p}$ and $\mathbf{m}$ are neither *conserved* nor *invariant*, and hence the time dependence in Eqs. (14) and (15) is crucial—and it is in no sense an "obvious" assumption. However, none of these questions is actually relevant for our purposes, since we take the dipole moments to be given.

[19]The fields $\mathbf{E}$ and $\mathbf{B}$ in Eqs. (21) and (22) carry additional delta-function terms. These delta terms do not contribute to the fields away from the site of the dipole. Since we are mainly interested in the radiation fields, these "contact" terms are not really relevant for us, and therefore they have been omitted [and in consequence the results are the same for "ampère" dipoles (current loops) and "Gilbert" dipoles (separated monopoles)].

[20]This change is feasible because Eqs. (24) and (25) do not require any spatial differentiation so that the coordinates $\mathbf{r}$ can be considered as vectorial parameters in such equations (see Ref. 4).

[21]The present author has not investigated to what extent Ellis' formulas in Ref. 9 for the radiation fields of the moving dipole are equivalent to those presented here.

[22]See, e.g., David J. Griffiths, *Introduction to Electrodynamics*, 2nd. ed. (Prentice–Hall, Englewood Cliffs, NJ, 1989), Eqs. (9.72) and (9.73).