A simple calculation of the rate of emission of energy and of linear and angular momentum by a point charge in arbitrary motion

R. Napolitano[a)] and S. Ragusa
Instituto de Física de São Carlos, Universidade de São Paulo, São Carlos, São Paulo 13560-970, Brazil

(Received 2 April 1998; accepted 17 March 1999)

We calculate directly the rates of emission of energy and linear momentum by a point charge in arbitrary motion using mathematical results that render the calculation of solid-angle integrals very simple. We show that the results of these explicit calculations agree with those based on covariance, illustrating how deeply special relativity is rooted in classical electrodynamics. Then, we use these covariance arguments to calculate the emission of angular momentum, which is a new result. We also indicate how this calculation can be done in the direct, but much longer, way.

I. INTRODUCTION

We have found that emission of linear and angular momentum by a multipolar charge distribution has received careful attention in the literature.[1(a),1(b),2(a),3(a)] Although there are very well-presented calculations of energy emission by an arbitrarily moving point charge,[1–4] the specialized textbooks do not present any material on the emission of angular momentum by the moving charge. The purpose of this article is twofold. First, it is meant to supplement classical electromagnetism textbooks by presenting a very simple calculation of the emission of energy and of linear momentum by an arbitrarily moving point charge, in a noncovariant formalism. The second aim of this paper is to present the calculation of the emission of angular momentum by the charge, which is a new result.

We do the calculation of the emission of energy and of linear momentum by an arbitrarily moving point charge without relying on cumbersome brute-force computations, or on relativistic arguments that could seem too obscure for an undergraduate student not accustomed to the intricacies of the covariant formulation of electrodynamics. The method we adopt is not difficult to understand and dramatically simplifies the calculations, so that any student can obtain general results without going through many hours of tedious work. Afterwards, we sketch the covariance arguments leading to the same result for the linear momentum emission rate. Then we reverse the procedure and calculate the angular momentum emission by using the experience gained through the covariance reasoning. We also indicate how the same result can be calculated in the direct way.

This work has pedagogical purposes and, accordingly, is organized as follows. In Sec. II we review the concept of emission of energy by an arbitrarily moving point charge. We employ Poynting's vector and the electric and magnetic fields in the radiation region to write down the rate of energy emission in terms of solid-angle integrals. These integrals are solved in Sec. III, using simplifying mathematical results. Section IV reviews the formulation of the rate of linear momentum emission by an arbitrarily moving point charge and shows how this quantity can be easily obtained using the results of Sec. III. In Sec. V we sketch the covariance arguments to obtain the linear momentum emission from its value at small velocities. A succinct version of this topic can be found in Ref. 3(b). In Sec. VI we calculate the emission of angular momentum by using covariance arguments, and we also indicate how the same result can be obtained through the direct, but much longer, method of Sec. III. Finally, in Sec. VII we conclude by summarizing the results of this work.

II. THE RATE OF ENERGY EMISSION BY A POINT CHARGE IN ARBITRARY MOTION

As explained in an exemplary way by Panofsky and Phillips,[4(a)] the rate of energy emitted by a charged particle at position $\mathbf{x}_c(t')$ is given, in Gaussian units, by

$$\frac{dW}{dt'} = \oint_S \left(\frac{dt}{dt'}\right)\left(\frac{c\mathbf{E}\times\mathbf{B}}{4\pi}\right)\cdot\mathbf{n}\,dS \tag{1}$$

(the loss would be $-dW/dt'$), where $c(\mathbf{E}\times\mathbf{B})/(4\pi)$ is Poynting's vector, $\mathbf{E}$ and $\mathbf{B}$ are the electric and magnetic fields, respectively, S is an arbitrary closed surface containing the point charge, $\mathbf{n}(|\mathbf{n}|=1)$ is the external normal to the surface S, t is the time at the observation point $\mathbf{x}$ on S, the particle's own time t' is related to t by $t'=t-r/c$, and $r=|\mathbf{x}-\mathbf{x}_c(t')|$. Since the energy emitted by the particle eventually reaches infinity, we choose S to be a spherical surface of very large radius r, centered at the position the particle occupied at its own time t', retarded with respect to t. The quantity dt/dt' is easily calculated to be

$$\frac{dt}{dt'} = 1-\mathbf{n}\cdot\boldsymbol{\beta}, \tag{2}$$

where $\boldsymbol{\beta}=\mathbf{v}/c$ and $\mathbf{v}=\mathbf{v}(t')$ is the velocity of the point charge at time t'. Let us notice that our choice of the surface S implies $\mathbf{n}=\mathbf{r}/r$, where $\mathbf{r}$ is the vector from the position of the charge at time t' to the point on the surface S where $\mathbf{n}$ is being calculated $[\mathbf{r}=\mathbf{x}-\mathbf{x}_c(t')]$.

Only the terms of first order in r^{-1} are necessary for the fields in Eq. (1), because Poynting's vector is quadratic in the fields and dS is proportional to r^2. Anyway, we will write the full expression of the electric field,[4(b)] because we will need it in Sec. VI for the calculation of the angular momentum emission:

$$\mathbf{E} = \frac{e(1-\beta^2)(\mathbf{n}-\boldsymbol{\beta})}{r^2(1-\mathbf{n}\cdot\boldsymbol{\beta})^3} + \frac{e\mathbf{n}\times[(\mathbf{n}-\beta)\times\mathbf{a}]}{rc^2(1-\mathbf{n}\cdot\boldsymbol{\beta})^3}, \tag{3a}$$

$$\mathbf{B} = \mathbf{n}\times\mathbf{E}, \tag{3b}$$

where $\mathbf{a}=\mathbf{a}(t')$ is the acceleration of the point charge at time t'. In the radiation zone, only the second term on the right-hand side of Eq. (3a) contributes. It follows from Eq. (3) that, in this region, $\mathbf{E}\cdot\mathbf{n}=0$ and $(\mathbf{E}\times\mathbf{B})\cdot\mathbf{n}=|\mathbf{E}|^2$. Then, together with Eq. (2) and the relation $dS=r^2 d\Omega$, Eq. (1) becomes[4(c)]

$$\frac{dW}{dt'}=\frac{e^2}{4\pi c^3}\int\frac{\{\mathbf{n}\times[(\mathbf{n}-\beta)\times\mathbf{a}]\}^2}{(1-\mathbf{n}\cdot\boldsymbol{\beta})^5}\,d\Omega. \tag{4}$$

As indicated in Ref. 4, the calculation of the integral through the analysis of the nodal lines is somewhat complicated. At this point we emphasize that there are two simple and straightforward ways to perform the angular integration. One of them is based on the simple fact that, when we expand the numerator, the integrals containing components n_i of the normal vector can be written as derivatives with respect to β_i of a single elementary Ω integral. The other is based on vector-algebra arguments to get the general form of these integrals. These simple procedures are carried out in detail in Sec. III.

III. THE CALCULATION OF THE SOLID-ANGLE INTEGRALS

There are probably many different ways to calculate solid-angle integrals involving components of the unit vector normal to the spherical surface, multiplied by some power of $(1-\mathbf{n}\cdot\boldsymbol{\beta})^{-1}$. In particular, Konopinski[3(c)] suggested in problems that Eq. (4) can be manipulated algebraically to be written as a linear combination of integrals of $(1-\mathbf{n}\cdot\boldsymbol{\beta})^{-s}$, with $s=3,4,5$, which can be easily integrated. Here we present a different and somewhat easier method, by means of which only one integral needs to be performed: the integral of $(1-\mathbf{n}\cdot\boldsymbol{\beta})^{-3}$.

Expanding the numerator of the integrand in Eq. (4) allows us to write

$$\frac{4\pi c^3}{e^2}\frac{dW}{dt'}=\mathbf{a}^2 I+2(\mathbf{a}\cdot\boldsymbol{\beta})a_iJ_i-(1-\beta^2)a_ia_jK_{ij}, \tag{5}$$

where henceforth we assume that repeated Roman indices are summed from 1 to 3 and

$$I=\int\frac{d\Omega}{(1-\mathbf{n}\cdot\boldsymbol{\beta})^3}=\frac{4\pi}{(1-\beta^2)^2}, \tag{6a}$$

$$J_i=\int\frac{n_i d\Omega}{(1-\mathbf{n}\cdot\boldsymbol{\beta})^4}=\frac{1}{3}\frac{\partial I}{\partial\beta_i}, \tag{6b}$$

$$K_{ij}=\int\frac{n_in_j d\Omega}{(1-\mathbf{n}\cdot\boldsymbol{\beta})^5}=\frac{1}{12}\frac{\partial^2 I}{\partial\beta_i\partial\beta_j}, \tag{6c}$$

where I is the only integral we had to calculate. Its value comes from choosing the z axis along β and writing $\mathbf{n}\cdot\boldsymbol{\beta}=\beta\cos\theta$ and $d\Omega=-d(\cos\theta)d\varphi$. Next, by using $\partial\beta^2/\partial\beta_i=2\beta_i$ we immediately get, from Eq. (6),

$$J_i=\frac{16\pi\beta_i}{3(1-\beta^2)^3}, \tag{7a}$$

$$K_{ij}=\frac{4\pi\left(\delta_{ij}+\dfrac{6\beta_i\beta_j}{1-\beta^2}\right)}{3(1-\beta^2)^3}. \tag{7b}$$

A second reasoning for the straightforward calculation of these integrals is as follows. Consider first J_i in Eq. (6b). Being a vector that depends only on $\boldsymbol{\beta}$, its general form must be $J_i=f(\beta^2)\beta_i$, where we can already guess that f will depend only on $\boldsymbol{\beta}^2$. To obtain f we just contract Eq. (6b) with $\boldsymbol{\beta}_i$ to reduce the integral to an elementary one. We get with $x=\beta\cos\theta$ and $d\Omega=-\beta^{-1}\,dx\,d\varphi$,

$$\frac{2\pi}{\beta}\int_{-\beta}^{\beta}\frac{x\,dx}{(1-x)^4}=f\beta^2.$$

After calculating this simple integral we immediately get Eq. (7a). Next, because K_{ij} in Eq. (6c) is a symmetric tensor that depends only on $\boldsymbol{\beta}$, its general form must be $K_{ij}=a\delta_{ij}+b\beta_i\beta_j$, where, again, the coefficients depend only on β^2. Contracting first i and j and then with β_i and β_j, we end up with two elementary integrals to determine the coefficients. We get

$$\begin{aligned}\frac{2\pi}{\beta}\int_{-\beta}^{\beta}\frac{dx}{(1-x)^5}&=3a+b\beta^2,\\ \frac{2\pi}{\beta}\int_{-\beta}^{\beta}\frac{x^2\,dx}{(1-x)^5}&=a\beta^2+b\beta^4.\end{aligned} \tag{8}$$

After solving these simple integrals we get Eq. (7b). From Eqs. (5) to (7), we finally obtain the energy emission of a moving point charge and the result can be expressed as[4(c)]

$$\frac{dW}{dt'}=\frac{2e^2}{3c^3}\left[\frac{\mathbf{a}^2-(\mathbf{a}\times\boldsymbol{\beta})^2}{(1-\beta^2)^3}\right]. \tag{9}$$

This is the instantaneous rate of radiation evaluated at the particle's time t'.

IV. THE LINEAR MOMENTUM EMISSION BY A POINT CHARGE IN ARBITRARY MOTION

In this section we illustrate how equally easy it is to calculate the linear momentum emission rate by a point charge in an arbitrary motion. By the method of Sec. III, only one Ω integral has to be performed, the one of $(1-\beta^2)^{-2}$. Analogously to the case of energy emission, we can express the rate of momentum emission by the charge as[1(a)]

$$\frac{dP_i}{dt'}=-\oint_S\left(\frac{dt}{dt'}\right)T_{ij}n_j\,dS \tag{10}$$

(the loss would be $-dP_i/dt'$, which is equal to the force reacting on the system), where P_i is the ith component of the momentum being emitted and T_{ij} is Maxwell's stress tensor,[4(d)]

$$T_{ij}=\frac{1}{4\pi}\left[E_iE_j+B_iB_j-\frac{1}{2}(\mathbf{E}^2+\mathbf{B}^2)\delta_{ij}\right]. \tag{11}$$

It follows from Eq. (3) that, in the radiation region, $\mathbf{E}\cdot\mathbf{n}=\mathbf{B}\cdot\mathbf{n}=0$ and, since $\mathbf{E}\perp\mathbf{B}$, then $\mathbf{E}^2=\mathbf{B}^2$. Hence, Eq. (11) gives

$$T_{ij}n_j=-\frac{1}{4\pi}\mathbf{E}^2 n_i. \tag{12}$$

Using Eqs. (2), (3a), and (12), we can write Eq. (10) as

$$\frac{4\pi c^4}{e^2}\frac{dP_i}{dt'}=\mathbf{a}^2A_i+2(\mathbf{a}\cdot\boldsymbol{\beta})a_jB_{ij}-(1-\beta^2)a_ja_kL_{ijk}, \tag{13}$$

where, with the indication

$$M=\int\frac{d\Omega}{(1-\mathbf{n}\cdot\boldsymbol{\beta})^2}=\frac{4\pi}{(1-\beta^2)},$$

we have

$$A_i=\int\frac{n_i d\Omega}{(1-\mathbf{n}\cdot\boldsymbol{\beta})^3}=\frac{1}{2}\frac{\partial M}{\partial\beta_i}=\frac{4\pi\beta_i}{3(1-\beta^2)^2}, \tag{14a}$$

$$B_{ij}=\int\frac{n_in_j d\Omega}{(1-\mathbf{n}\cdot\boldsymbol{\beta})^4}=\frac{1}{6}\frac{\partial^2 M}{\partial\beta_i\partial\beta_j}=\frac{4\pi\left(\delta_{ij}+\dfrac{4\beta_i\beta_j}{1-\beta^2}\right)}{3(1-\beta^2)^2}, \tag{14b}$$

$$\begin{aligned}L_{ijk}&=\int\frac{n_in_jn_k d\Omega}{(1-\mathbf{n}\cdot\boldsymbol{\beta})^5}\\&=\frac{1}{24}\frac{\partial^3 M}{\partial\beta_i\partial\beta_j\partial\beta_k}\\&=\frac{4\pi\left(\delta_{ij}\beta_k+\delta_{ik}\beta_j+\delta_{jk}\beta_i+\dfrac{6\beta_i\beta_j\beta_k}{1-\beta^2}\right)}{3(1-\beta^2)^3}.\end{aligned} \tag{14c}$$

The quantities A_i and B_{ij} are obtained in a way analogous to the calculation leading to Eq. (7). Another way of getting the result for L_{ijk} is to notice that Eq. (14c) is a completely symmetric third-order tensor (changes sign under inversion) depending only on $\boldsymbol{\beta}$. Therefore, its general form is $L_{ijk}=A(\delta_{ij}\beta_k+\delta_{ik}\beta_j+\delta_{jk}\beta_i)+B\beta_i\beta_j\beta_k$, containing only odd powers of β_i. Contraction with $\delta_{ij}\beta_k$ and with $\beta_i\beta_j\beta_k$ leads again to elementary integrals for the determination of A and B, and the final result is again Eq. (14c).

Finally, from Eqs. (13) and (14), it readily follows that

$$\frac{d\mathbf{P}}{dt'}=\frac{2e^2}{3c^4}\left[\frac{\mathbf{a}^2-(\mathbf{a}\times\boldsymbol{\beta})^2}{(1-\beta^2)^3}\right]\boldsymbol{\beta}. \tag{15}$$

This result was first obtained by Abraham[5] by a different, and rather involved, method. We stress that this expression is the instantaneous rate of momentum emission evaluated at the particle's time t'.

V. COVARIANCE ARGUMENTS

In this section we analyze the previous result in the light of covariance arguments of special relativity, obtaining the rate of linear momentum emission from its value at small velocities. Essentially, here we present an expanded variation of the succinct discussion by Konopinski[3(b)] and, following this section, we employ the same covariance reasoning to calculate the emission of angular momentum.

After a rather simple calculation we obtain the ith component of the rate of momentum emission at small velocities:

$$\left(\frac{dP_i}{dt'}\right)_1=-\frac{2e^2}{3c^4}\mathbf{a}^2\beta_i, \tag{16}$$

where the index 1 means that this value is of first order in β. To get this result we just expand $(1-\mathbf{n}\cdot\boldsymbol{\beta})^{-5}\approx 1+5\mathbf{n}\cdot\boldsymbol{\beta}$ in the integrand of Eq. (10), use the known results[1(c)]

$$\int n_in_j\, d\Omega=(4\pi/3)\delta_{ij},$$

$$\int n_in_jn_kn_l\, d\Omega=(4\pi/15)(\delta_{ij}\delta_{kl}+\delta_{ik}\delta_{jl}+\delta_{il}\delta_{jk}),$$

which actually inspired the second method to obtain the Ω integrals in Sec. III, and notice that the integral over an odd number of normal components is equal to zero.

Equation (16) holds in a reference frame where the magnitude of the velocity of the particle is much less than the speed of light. Now it is straightforward to obtain a covariant expression whose purely spatial components reduce to Eq. (16) in the limit of low speed:

$$\frac{dP^\mu}{d\tau}=\frac{2e^2}{3c^4}\alpha^\nu\alpha_\nu u^\mu, \tag{17}$$

where henceforth we assume that repeated Greek indices are summed from 0 to 3, $\alpha^\mu=du^\mu/d\tau$ is the four-acceleration, $u^\mu=dx^\mu/d\tau$ is the four-velocity, τ is the proper time, $x^0=ct'$, $x^1=x_c(t')$, $x^2=y_c(t')$, $x^3=z_c(t')$, $x_0=ct'$, $x_1=-x_c(t')$, $x_2=-y_c(t')$, and $x_3=-z_c(t')$. Therefore,

$$\alpha^\mu\alpha_\mu=-\left[\frac{\mathbf{a}^2-(\mathbf{a}\times\boldsymbol{\beta})^2}{(1-\beta)^3}\right]. \tag{18}$$

As this result reduces to $-\mathbf{a}^2$ and τ to t' when $\beta=0$, and u_i reduces to $c\beta_i$ to first order in β, the space part of Eq. (17) reduces to Eq. (16) and, therefore, this covariant expression should be the desired result. Equation (17) is valid in any other frame, that is, for arbitrary velocities. In terms of $\boldsymbol{\beta}$ and $\mathbf{a}$, the space part of Eq. (17) becomes Eq. (15) if we consider Eq. (18). This is the method we shall use in Sec. VI to calculate the emission of angular momentum.

VI. EMISSION OF ANGULAR MOMENTUM BY AN ARBITRARILY MOVING POINT CHARGE

In this section we calculate the emission of angular momentum for the arbitrary motion of the particle, which is a new result, by using covariance arguments. For this purpose, we calculate the emission for low velocities. Thence we boost the answer to arbitrary velocities by using the experience gained in Sec. V. Again, on pedagogical grounds, we indicate afterwards how the calculation could be performed by the direct method of Sec. III.

The emission of angular momentum is given by[2(b)]

$$\frac{dL_i}{dt'}=\oint_S\left(\frac{dt}{dt'}\right)n_jM_{ji}\,dS,$$

where

$$M_{ji}=\epsilon_{imn}T_{jm}x_n,$$

T_{jm} is given by Eq. (11), and dt/dt' is given by Eq. (2). Therefore, taking into account that $\mathbf{n}\cdot\mathbf{B}=0$, from Eq. (3b), we get

$$\frac{d\mathbf{L}}{dt'}=\frac{1}{4\pi}\oint_S\left(\frac{dt}{dt'}\right)(\mathbf{n}\cdot\mathbf{E})(\mathbf{E}\times\mathbf{n})r^3\,d\Omega. \tag{19}$$

For the calculation to first order in $\boldsymbol{\beta}$, the electric field of Eq. (3a) becomes

$$\mathbf{E}=\frac{e(\mathbf{n}-\boldsymbol{\beta})}{r^2(1-\mathbf{n}\cdot\boldsymbol{\beta})^3}+\frac{e\mathbf{n}\times[(\mathbf{n}-\beta)\times\mathbf{a}]}{rc^2(1-\mathbf{n}\cdot\boldsymbol{\beta})^3}. \tag{20}$$

We could expand the denominators, but it is more convenient to do so after multiplication by dt/dt', given by Eq. (2). When calculating $\mathbf{n}\cdot\mathbf{E}$, only the first, r^{-2}, term does not vanish. On the other hand, when calculating $\mathbf{n}\times\mathbf{E}$, only the second, r^{-1}, term contributes to the integral, because of the factor r^3 in Eq. (19). A straightforward calculation gives

$$\frac{d\mathbf{L}}{dt'}=\frac{e^2}{4\pi c^2}\oint_S[3(\mathbf{n}\times\mathbf{a})(\mathbf{n}\cdot\boldsymbol{\beta})+(\mathbf{n}\times\boldsymbol{\beta})(\mathbf{n}\cdot\mathbf{a})]d\Omega,$$

which can also be expressed in terms of Cartesian components as

$$\frac{dL_i}{dt'}=\frac{e^2}{4\pi c^2}[3\epsilon_{ijk}a_k\beta_l+\epsilon_{ijk}a_l\beta_k]\oint_S n_jn_l\,d\Omega$$
$$=\frac{2e^2}{3c^2}\epsilon_{ijk}\beta_ja_k, \tag{21}$$

where we have used Eq. (7b) with $\boldsymbol{\beta}=0$. To make the generalization to the covariant form of Eq. (21), it is convenient to eliminate the ϵ_{ijk} tensor by defining the quantity

$$\frac{d\Lambda_{mn}}{dt'}=\epsilon_{mni}\frac{dL_i}{dt'}=\frac{2e^2}{3c^3}(v_ma_n-v_na_m). \tag{22}$$

Now it is straightforward to obtain a covariant expression whose purely spatial components reduce to Eq. (22) in the limit of low speed:

$$\frac{d\Lambda^{\mu\nu}}{d\tau}=\frac{2e^2}{3c^3}(u^\mu\alpha^\nu-u^\nu\alpha^\mu), \tag{23}$$

where the four-velocity u^μ and four-acceleration α^μ are defined in Sec. V. In terms of $\boldsymbol{\beta}$ and of $d\boldsymbol{\beta}/dt'$, the space part of Eq. (23) becomes

$$\frac{d\Lambda_{ij}}{d\tau}=\frac{2e^2}{3c(1-\beta^2)^{3/2}}\left[\beta_i\frac{d\beta_j}{dt'}-\beta_j\frac{d\beta_i}{dt'}\right],$$

or

$$\frac{d\Lambda_{ij}}{dt'}=\frac{2e^2}{3c^2(1-\beta^2)}[\beta_ia_j-\beta_ja_i].$$

Hence, from Eq. (22), we obtain

$$\frac{dL_i}{dt'}=\frac{1}{2}\epsilon_{imn}\frac{d\Lambda_{mn}}{dt'}=\frac{2e^2}{3c^2(1-\beta^2)}\epsilon_{imn}\beta_ma_n,$$

or

$$\frac{d\mathbf{L}}{dt'}=\frac{2e^2}{3c^2}\frac{\boldsymbol{\beta}\times\mathbf{a}}{(1-\beta^2)}, \tag{24}$$

which is the new result. It is the instantaneous rate of emission of angular momentum at the particle's time t'. In the case of a charge in a central-force motion, this result reduces to Eq. (16.16) of Ref. 2 if we take the limit of small velocities [see also Refs. 2(a) and 3(d)].

Just for completeness, we indicate how the calculation can be done by the direct method of Sec. III. From Eqs. (3a) and (19), it follows that we should calculate the solid-angle integral

$$\frac{d\mathbf{L}}{dt'}=\frac{e^2(1-\beta^2)}{4\pi c^2}\int\left[\frac{(\mathbf{n}\cdot\mathbf{a})(\mathbf{n}\times\boldsymbol{\beta})}{(1-\mathbf{n}\cdot\boldsymbol{\beta})^4}+\frac{\mathbf{n}\times\mathbf{a}}{(1-\mathbf{n}\cdot\boldsymbol{\beta})^3}\right]d\Omega. \tag{25}$$

If we write Eq. (25) in terms of its components, we see that substituting the integrals A_i and B_{ij} of Eqs. (14a) and (14b) into (25) gives, after some algebraic operations, the result of Eq. (24).

VII. CONCLUSION

In this paper we have shown two straightforward methods to calculate the solid-angle integrals for both the energy and linear momentum emission rates by a charged point particle. Because the emission rate of linear momentum is an important feature of accelerated charges, we believe that this work really supplements the existing material in classical textbooks. We have also shown how covariance arguments can further reduce the calculation.

In a second stage we have calculated the emission of angular momentum by the moving point charge, which constitutes a new result. This we have done reversing the procedure, that is, using the covariance arguments. We have also indicated how it could be obtained through the direct, although rather long, process of calculation.

[a]Electronic mail: reginald@if.sc.usp.br

[1]L. D. Landau and E. M. Lifshitz, *The Classical Theory of Fields* (Pergamon, New York, 1975), 4th ed.; (a) p. 190; (b) pp. 193–194; (c) p. 189.

[2]J. D. Jackson, *Classical Electrodynamics* (Wiley, New York, 1998), third edition; (a) pp. 451–452, Problems 9.8 and 9.9; (b) pp. 658–660.

[3]E. J. Konopinski, *Electromagnetic Fields and Relativistic Particles* (McGraw–Hill, New York, 1981); (a) p. 226; (b) p. 440; (c) p. 321, Problems 10.5 and 10.6; (d) p. 460.

[4]W. K. H. Panofsky and M. Phillips, *Classical Electricity and Magnetism* (Addition–Wesley, Reading, MA, 1962), 2nd ed.; (a) p. 360; (b) p. 356; (c) p. 370; (d) p. 181.

[5]M. Abraham, ''Prinzipien der Dynamik des Elektrons,'' Ann. Phys. (Leipzig) **10**, 105–179 (1903).

IMMORTALITY

So Greek mathematics is 'permanent', more permanent even than Greek literature. Archimedes will be remembered when Aeschylus is forgotten, because languages die and mathematical ideas do not. ''Immortality' may be a silly word, but probably a mathematician has the best chance of whatever it may mean.

G. H. Hardy, *A Mathematician's Apology* (Cambridge University Press, 1969; reprint of 1940 edition), p. 81.