the extremely small area involved, this has no appreciable effect whatever on the total luminosity of the nebula.) Reduced with the aid of the other Cambridge observations, the Mt. Wilson surface brightness at the centre is then $14^{m}\cdot 8/(\text{second of arc})^2$. On account of finite resolving power this is a minimum value, representing the least concentration which must really occur at the centre. We understand that unpublished Mt. Wilson material shows that the spectrum of the nucleus is of a peculiar dwarf A type. It seems likely that this is mainly produced by dwarf A stars, and if the average absolute magnitude of these is $+1$, the star density at the centre attains at least $1000/(\text{parsec})^3$.* There is, however, a very steep fall of density outwards, and at $0'\cdot 1$, *i.e.* about 7 parsecs from the centre, where the surface brightness is $17^{m}\cdot 2/(\text{second of arc})^2$, the density cannot well exceed 100 stars/(parsec)^3.

The centre of M 33 is quite different in character from that of M 31. There is no gradual increase of luminosity to a quasi-stellar nucleus. The spiral arms spring from a hazy mass which is of roughly constant surface brightness and has a diameter about $1'\cdot 5$. Lying along the spiral arms there are large numbers of stellar or quasi-stellar objects, apparently belonging to the nebula, and some also are to be found in the nebulous centre just mentioned. A few of the stars in the immediate neighbourhood of the centre may be foreground objects. The Cambridge reflector is unable to reveal any detail which can be definitely assigned as the nucleus of the nebula and which at the same time has enough area for the surface brightness to be measured. Excluding stellar objects, the maximum surface brightness of the "hazy mass", measured with one standard star exposure (Selected Area 46), is about $20^{m}\cdot 3/(\text{second of arc})^2$.

DYNAMICAL EFFECTS OF RADIATION IN THE SOLAR SYSTEM.

H. P. Robertson.

(Communicated by Professor H. N. Russell)

1. *Introduction.*—In connection with his investigations on radiation pressure, J. H. Poynting † at one time considered the consequences of absorption and re-emission of solar radiation for the motion of small bodies. Of these, perhaps the most important for cosmogony was a tangential drag, which would give rise to a decrease in the angular momentum of the particle, and thus cause it eventually to fall into the Sun. This drag Poynting attributed to a back-pressure of radiation tending to retard the motion through the ether of the emitting body, which he conceived as due to a

* Sinclair Smith (*Ap. J.*, **82**, 192, 1935) has deduced 8·8 10^5 stars/(parsec)^3 for the centre of the companion nebula, N.G.C. 221.

† *Phil. Trans. Roy. Soc.*, A, **202**, 525, 1903; reprinted, with correction of numerical slips, as Art. 20, p. 304, of his *Collected Scientific Papers*, Cambridge, 1920. Reference may also be made to these collected works for a number of Poynting's addresses dealing with the same subject, notably Arts. 22, 64–67.

crowding-up of radiation in front of the particle and to a corresponding thinning-out behind; the magnitude of the resulting force (as corrected in a note at the end of the paper referred to above) was given as $\frac{1}{3}Rv/c^2$, where R is the rate at which the particle is radiating energy, v is its velocity through the ether, and c is the velocity of light. Now, since the Sun's gravitational effect on bodies of equal density goes down with the cube of their linear dimensions, whereas the rate at which they receive and re-radiate energy decreases only with the square of these dimensions, the effect in question should be the more marked the smaller the body. Thus, according to Poynting, a spherical body of radius 1 cm. and density 5·5, starting from the Earth's distance from the Sun and with the Earth's orbital velocity, could make at most some 10^8 revolutions before falling into the Sun, while one of radius 10^{-3} cm. could survive at most some 10^5 revolutions.

Although Poynting emphasized the cosmogonic implications of these results, they do not seem to have been widely recognized at the time.* The subject was reopened some years later by J. Larmor, in an address before the International Congress of Mathematicians at Cambridge, in which he gave an alternative approximate treatment, on classical electromagnetic theory, of "the retarding force exerted on a body translated through the æther with uniform velocity [v], arising from its own radiation," and showed that, to terms of first order in v, the force in question was given by Rv/c^2 †—three times Poynting's corrected value.

But that this force could by itself act as a brake on the movement of the particle is clearly in contradiction with the theory of relativity,‡ which is in turn consistent with the electromagnetic theory on which the result was derived. This circumstance led L. Page to undertake a detailed examination of the matter, and he was indeed able to show that, to terms of the order in question, the velocity of a moving body does *not* suffer a retardation in consequence of its own radiation alone.§ That this is in fact the case was subsequently admitted by Larmor in a Postscript to Poynting's *Collected Papers*, where he states (p. 757) that "the remarkable result seems to be established that an isolated body cooling in the depths of space would not change its velocity through the æther, the retardation due to the back thrust of radiation issuing from it being just compensated by increase of velocity due to momentum conserved with diminishing mass." And here, for the first time, the true sufficient cause of the drag in question was clearly stated, and the paradox resolved; in Larmor's words, "But for Poynting's particle

* An interesting application of these notions is to be found, however, in the attempt of H. C. Plummer, *M.N.*, **65**, 229, 1905; **67**, 63, 1906, to explain in this way the anomaly in the mean motion of Encke's comet.

† *Proc. 5th Internat. Cong. Math.*, **1**, 197, Cambridge, 1913; reprinted in part in Poynting's *Collected Papers*, p. 426. The result quoted above is that obtained on p. 215 of the *Proceedings*.

‡ As was pointed out in *The Observatory*, **40**, 278, 1917, in a review of some related work, to which Larmor replied in a letter to *Nature*, **99**, 404, 1917, insisting on the reality of the retardation.

§ *Phys. Rev.*, **11**, 376, 1918; **12**, 371, 1918; summary of first article in *Proc. Nat. Acad. Sci.*, **4**, 47, 1918.

describing a planetary orbit the radiation from the Sun comes in, which restores the energy lost by radiation from the particle, and so establishes again the retarding force $[-Rv/c^2]$." However, Larmor goes further and asserts that the particle is subject to yet another drag, of the same amount, due to the "astronomical aberration of light," thus doubling the result which he had previously obtained. Closer examination shows, however, that this doubling—which, in its entirety, would again lead into contradiction with the theory of relativity—affects only the radial component of the drag, to terms of the order retained by Larmor, and is therefore ineffective in producing a further decrease of angular momentum.

At the instance of Professor H. N. Russell, who called my attention to Poynting's paper, I have recently undertaken an examination of the whole question from the standpoint of the theory of relativity. On communicating my findings to him, he has urged that I make them public ; this, then, is the purpose of the present note. In the following Section the exact relativistic equations are derived, and in subsequent Sections are applied (to terms of first order in v) to those questions which first attracted Poynting's attention. I may say that for an understanding of these latter applications, the reader unfamiliar with the notation and methods of the theory of relativity need not have read the more rigorous derivation of Section 2, for to that approximation, in which only linear terms are preserved, the equations of motion are readily interpretable in classical terms, as set forth at the opening of Section 3. Aside from its intrinsic interest, the precise relativistic treatment was deemed necessary to establish beyond all cavil the existence of just those linear terms, and no others, which are employed in the subsequent development. It is a pleasure to record here my indebtedness to Professor Russell for his stimulating comments and advice, particularly on the astronomical aspects of the subject, to which the latter Sections are devoted.

2. *Relativistic Equations of Motion.*—We propose, then, to investigate the motion of a small, intrinsically spherical body in a beam of radiation, under the supposition that it absorbs all the energy falling upon it, and that it re-emits this energy at the same rate isotropically with respect to a reference system in which it is instantaneously at rest. This latter circumstance suggests that we first set up the equations in the rest system, and then transform them back into the original inertial system ; this procedure will have the additional advantage that we may take over, without essential modification, results on radiation pressure which have been established by purely classical methods for bodies at rest. As will be seen in the sequel, it will suffice to consider the problem in the frame of the special theory of relativity, for the modifications necessitated by the general theory can most easily be made later, in case they are desired.

Let the original reference system S, which may in the applications be taken as one in which the Sun is instantaneously at rest, be defined by a Minkowski co-ordinate system $x^\mu=(t, x, y, z)$, in which the invariant line element assumes the canonical form

$$ds^2=g_{\mu\nu}dx^\mu dx^\nu=dt^2-(dx^2+dy^2+dz^2)/c^2, \qquad (2.1)$$

where the repeated indices μ, ν are summed over their range 0, 1, 2, 3. The rotational motion of the body is here ignored, as its energetic effects are presumably negligible and we are assuming that in all cases the mechanism of transfer of energy within the body is such that it is ultimately radiated isotropically from all portions of the surface of the body; under these conditions the motion of the body may be specified by a unit four-vector of velocity u^μ, satisfying the equation

$$g_{\mu\nu}u^\mu u^\nu = 1, \tag{2.2}$$

or by its energy-momentum vector mu^μ, where m is its proper mass. For considerations not involving diffraction or selective pressure—the influence of which is touched upon later—the primary radiation may be thought of as a plane-parallel beam, defined in direction by a null-vector l^μ whose components are chosen subject to the convenient normalization condition $l^0 = 1$ in the given reference system S:

$$g_{\mu\nu}l^\mu l^\nu = 0, \qquad l^0 = 1; \tag{2.3}$$

the density of energy in this incoming beam may then be taken as the temporal average d of the electromagnetic energy density. Under these circumstances the primary radiation is completely specified by the energy-momentum-stress tensor

$$e^{\mu\nu} = dl^\mu l^\nu, \tag{2.4}$$

representing a beam of energy density $e^{00} = d$ carrying momentum at the rate d per unit cross-section in the spatial direction defined by the components l^α ($\alpha = 1, 2, 3$) of l^μ, and exerting a pressure d in this direction.

In order to carry out the programme outlined above, we introduce, in the neighbourhood of an event E in the history of the particle, a rest system Σ_E, with co-ordinates $\xi^\mu = (\tau, \xi, \eta, \zeta)$. We denote the components in Σ_E of the vectors u^μ, l^μ, defining the motion of the particle and the direction of the beam, by the corresponding Greek letters v^μ, λ^μ. Then at E

$$v^\mu = \delta_0{}^\mu, \qquad \lambda^\alpha = c\lambda^0\nu^\alpha \qquad (\alpha = 1, 2, 3), \tag{2.5}$$

where the ν^α, defining the direction of the beam in the proper-space of the particle, may, in consequence of the fact that λ^μ is a null-vector, be considered as the three components of a unit vector in the sense of ordinary vector analysis; it is to be noted that we are here not free to choose λ^0 unity, for the normalization is completely given by the condition that the corresponding quantity l^0 in the reference system S be unity. The energy-momentum vector of the particle is here given by mv^μ, and the energy-momentum-stress tensor of the primary radiation by

$$\epsilon^{\mu\nu} = \delta\lambda^\mu\lambda^\nu/\lambda_0{}^2, \tag{2·6}$$

where δ is its energy density ϵ^{00} in the rest system Σ.

The assumption that the particle re-radiates energy at the same rate at which it receives it insures that the proper mass m of the particle is a constant of the motion, and this and the fact that v^μ is a unit vector together imply that the time rate of change of mv^0 vanishes at E. Now the proper rate of change of the spatial components mv^α of the energy-momentum vector is equal to the impressed force, and since the particle is assumed to possess spherical symmetry and to radiate isotropically in the rest system Σ, this force can only act in the direction ν^a of the spatial projection of the null-vector λ^μ of the incoming beam. Since the particle is instantaneously at rest in the system Σ the classical results on radiation pressure may be called upon for the determination of the magnitude ϕ of the force in question; we are at the moment primarily interested in the fact that it is in any case directly proportional to the energy density δ of the incoming beam—even where diffraction and selective pressure play an important rôle, although we may then be compelled to interpret much of what has gone before in a statistical sense. We may therefore write

$$\phi = A\delta, \tag{2.7}$$

where in the case of principal interest below, in which the radius a of the sphere is large compared with the wave-length of the radiation, the factor A is merely the projection πa^2 normal to the beam of the area of the illuminated hemisphere; we prefer, however, to retain the more non-committal formulation (2.7) to allow for cases in which diffraction or resonance may be thought of as giving rise to a larger *effective cross-section A*.*

The equations of motion thus obtained in the rest system Σ may now be brought together in the four-dimensional formulation

$$\frac{dmv^\mu}{d\tau} = \frac{\phi}{c\lambda^0}\{\lambda^\mu - \lambda^0 v^\mu\}; \tag{2.8}$$

for, on taking into account the relations (2.5), the temporal component $\mu = 0$ of the right-hand side is seen to vanish at E, thus expressing the conservation of mass, and the spatial component $\mu = a$ is equal to the component $\phi\nu^a$ of the force due to radiation pressure. The first term $\phi\lambda^\mu/c\lambda^0$ represents the total power-force four-vector due to the incident radiation, with which it coincides completely in direction, and the second that due to the radiation re-emitted at the rate ϕc.†

* Investigations on radiation pressure when diffraction is of importance are to be found in papers by K. Schwarzschild, *Sitzungsber. Bayer. Akad. d. Wiss.*, **31**, 293, 1901, and by P. Debye, *Ann. d. Phys.*, **30**, 57, 1909. The selective pressure effects due to resonance have also been treated by Debye, *loc. cit.*, by L. Page, *Astrophys. Journ.*, **52**, 65, 1920, by W. Baade and W. Pauli, *Naturwiss.*, **15**, 49, 1927, and by A. Unsöld, *Zeits. f. Phys.*, **44**, 793, 1927. The application of this latter effect to the form of the tail of a comet, together with references to the literature, have been given recently by K. Wurm, *Zeits. f. Astrophys.*, **10**, 285, 1935—where it is pointed out that the pressure on CO^+ due to the solar radiation is of the order of 100 times the gravitational attraction.

† It may be remarked, however, that the validity of these equations is not fully circumscribed by the particular scattering mechanism here contemplated. For if, for example, we consider the effect of the radiation on a small perfectly reflecting sphere, the equations of motion are exactly the same as those above for complete absorption and subsequent re-emission, the additional force on that region of the sphere in which the normal makes an angle of less than $\pi/4$ with the direction from which the beam is coming being just counterbalanced by an equal and opposite force on those portions in which the angle is between $\pi/4$ and $\pi/2$, as can be shown from Poynting's work on normal pressure and tangential stress due to radiation, *C.S.P.*, p. 318.

In order to obtain from (2.8) the equations of motion of the particle in terms of the original system S with co-ordinates x^μ, it is only necessary to remark that the λ^μ, v^μ appearing in (2.8) are the components of the two four-vectors which are represented in S by l^μ, u^μ, and to express in scalar form the quantities λ^0, ϕ occurring in the coefficients. With regard to λ^0, we note that in the rest system Σ it is the scalar product of λ^μ with v^μ, and is therefore to be replaced by the scalar

$$w = l_\mu u^\mu (= \lambda^0). \tag{2.9}$$

Next, we note that δ, which appears in the expression (2.7) for the proper-force ϕ, is simply the component ϵ^{00} of the energy tensor (2.6) in the rest system Σ, and may therefore be replaced by the double inner product $\epsilon^{\mu\nu} v_\mu v_\nu$ of this tensor with the velocity vector v^μ. But the value $e^{\mu\nu} u_\mu u_\nu$ of this scalar in terms of the system S is, by (2.4) and (2.9), simply $w^2 d$, and we may therefore write

$$\delta = w^2 d, \quad \text{or} \quad \phi = w^2 f, \tag{2.10}$$

where we have introduced for convenience the force $f = Ad$ which would act on the particle if it were at rest in the system S.

The equations of motion of the particle in the original reference system S are therefore

$$\frac{dmu^\mu}{ds} = \frac{fw}{c}\{l^\mu - wu^\mu\}, \tag{2.11}$$

where s is the proper time of the particle and, in recapitulation, $w = l_\mu u^\mu$, $l^0 = 1$, and f is the force which *would* act on the particle if it *were* at rest in the system S—and is carefully to be distinguished from the force which *does* act on it when viewed from the system Σ in which it *is* at rest.

The above development has been based on the special theory of relativity, and we have here to add a few words concerning the modifications required on going over to the general theory. If we consider the particle as a test particle, whose effect on the gravitational metric due to the Sun or other bodies is negligible, and ignore possible effects due to the interaction of radiation and gravitation in computing the impressed force f in normal co-ordinates at E, the only change required is the addition of the inertial terms

$$m\left\{ {\mu \atop \nu\sigma} \right\} u^\nu u^\sigma \tag{2.12}$$

on the left-hand side of the equations of motion (2.11), where the Christoffel symbols are computed from the given metric. This is equivalent to adding, in the classical approximation, the Newtonian gravitational force, as is done directly in the succeeding section, and, in the next approximation, the small relativistic correction which is responsible for the perihelion advance of a Keplerian orbit.

Before proceeding to the applications, let us examine the equations (2.11) with reference to the point which has caused so much confusion in the literature referred to in the introductory section. First, as the equations stand they must, of course, lead to the conservation of proper mass—as can be seen by multiplying through by u_μ and summing with respect to μ, taking into account the relations (2.2) and (2.9). Clearly the second term does here represent an actual retarding force, which must be combined with the force in the direction l^μ due to the radiation pressure of the incoming beam. But equally clearly this force would not give rise to a retardation of the velocity if the first term were left out of account, for we would then have

$$m\frac{du^\mu}{ds}+\frac{dm}{ds}u^\mu=-\frac{\phi}{c}u^\mu, \tag{2.13}$$

and since in this case the particle would be losing mass at the rate $\phi c/c^2$ at which energy is being radiated away (in mass units), the second term on the left just cancels that on the right, whence the velocity u^μ is constant. This is in fact the exact relativistic form of the explanation proposed by Larmor in the approximate treatment referred to in Section 1, and is in full agreement with the more detailed studies by Page. The explicit introduction of the radiation force, which represents the immediate source of the energy emitted, serves to remove the compensating term on the left in the above equation, and to restore the drag as giving rise to a true retardation of velocity. On the other hand, it is apparent that any effect attributable to aberration must already be included in the above treatment; its introduction can at most suggest an alternative interpretation of the results, and cannot lead to additional terms, as was held by Larmor.*

3. *Newtonian Approximation. Equations of Motion of Particle in Field of the Sun.*—For problems in which the specifically relativistic effects may be expected to play no important rôle, the equations derived in Section 2 above may be reduced to a simpler form, in which they are directly comparable with the approximate equations derived by Poynting and by Larmor. Here we are interested only in terms of first order in the ratio of the velocity of the particle to that of light, and we may accordingly revert to the Newtonian scheme, expressing the equations of motion in terms of the two vectors

$$v^a=\frac{dx^a}{dt}, \qquad n^a=l^a/c, \tag{3.1}$$

the second of which is the unit vector in the direction of the incident beam. The various quantities involved in the equations (2.11) may then be taken, to

* It may appear paradoxical that there can be a retardation transverse to the beam in one reference system and not in another, but it is to be remembered that the angle between the spatial projections of two four-vectors, such as l^μ and u^μ, is not invariant. Thus in the case in which the incoming beam is rigorously plane-parallel, the motion of the particle, as viewed from the standpoint of an inertial system Σ with respect to which it was at one time at rest, is always along the direction of the spatial projection l^a of the null-vector defining the beam, whereas from the standpoint of an arbitrary reference system S there will in general be a transverse component. The direct integration of the equations (2.11) for this case shows that the ratios of the velocity components u^μ approach those of the null-vector l^μ as $s\to\infty$, and that on following the motion backward in time these ratios approach those of the components k^μ of some other null-vector; the possible rest systems Σ are characterized geometrically by the fact that their time axes lie in the plane determined by the two null-vectors k^μ, l^μ.

terms of first order in v^α, as

$$u^0 = 1, \qquad u^\alpha = v^\alpha, \qquad w = 1 - v_n/c, \tag{3.2}$$

where v_n is the component of the velocity v^α in the direction n^α; the equations of motion then become, to this order,

$$m\frac{dv^\alpha}{dt} = f(1 - v_n/c)n^\alpha - fv^\alpha/c. \tag{3.3}$$

The first term on the right may be interpreted as that due directly to the radiation pressure, in the direction of the incoming beam but weakened by the Doppler factor $1 - v_n/c$, while the second represents the tangential drag in question. The total drag in the direction of the beam thus arises from both terms, and is in magnitude $2fv_n/c$, while that in directions transverse to the beam is only fv'/c, where v' is the component of v^α in this direction.*

These approximate equations of motion may be readily interpreted in terms of notions familiar in classical mechanics and electrodynamics. The first force term is, as mentioned above, merely that due to direct radiation pressure on a body whose velocity in the direction of the beam is v_n. Now this particle is absorbing energy at the rate $cf = cAd$, and is re-radiating it in such a way that the outgoing radiation may be thought of as carrying away electromagnetic momentum at the rate $(cf/c^2)v^\alpha$, since it is distributed isotropically about the particle, which is in turn moving with the velocity v^α. But in virtue of the general law of conservation of total (mechanical plus electromagnetic) momentum, this process must cause the particle to lose momentum at the rate fv^α/c, which therefore asserts itself as the retarding force represented by the last term in the equations (3.3).

In the case of a particle moving in the gravitational field of the Sun, absorbing and re-emitting the Sun's radiation falling on it in accordance with the above scheme, we must interpret n^α as the unit vector along the radius vector from the Sun, and take the energy density d as falling off inversely with the square of the distance from the Sun. Let S be the solar constant (radiation falling on a square centimetre normal to the Sun's rays at the mean distance b of the Earth $= 1.35 \times 10^6$ ergs per sec. per cm.^{2}); then at the distance r from the Sun

$$d = Sb^2/cr^2, \quad f = m\alpha c/r^2, \quad \text{where} \quad \alpha = \frac{ASb^2}{mc^2} = \frac{3Sb^2}{4a\rho c^2}\left(= \frac{2.51 \times 10^{11}}{a\rho}\right); \tag{3.4}$$

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* Although the additional drag in the direction of the beam is a straightforward consequence of the Doppler factor alluded to above, an effect found by Poynting in a note added to his principal paper on the present question, it seems to have been neglected in most of the treatments referred to in the introductory section. The only paper which has come to my attention in which it has been taken explicitly into account is the second of Plummer's papers on the motion of comets, in which the author thanks Poynting for having called it to his attention. It is conceivable that it is this effect which Larmor had in mind in referring to aberration (p. 756 of Poynting's *C.S.P.*—*cf.* Poynting's somewhat misleading use of the term "aberration," *loc. cit.*, p. 331), but it is to be emphasized that it affects only the component in the direction of the beam, and does not give rise in this approximation to an additional transverse drag.

the second expression for α being that for the case in which we are dealing with a small sphere of radius a, which is nevertheless large compared with the wave-length of the incident radiation, and of material density ρ.* Furthermore, we must of course add on to the right-hand side of (3.3) the force GMm/r^2, in the direction $-n^{\alpha}$, due to the Sun's gravitational attraction. These equations of motion, in polar co-ordinates (r, θ) in the plane of the orbit with pole at the Sun, are then readily found to be

$$\ddot{r} - r\dot{\theta}^2 = -\frac{\mu}{r^2} - \frac{2\alpha\dot{r}}{r^2}, \quad \frac{1}{r}\frac{d}{dt}(r^2\dot{\theta}) = -\frac{\alpha\dot{\theta}}{r}, \quad \text{where} \quad \mu = \mu_0 - \alpha c \tag{3.5}$$

and the dot indicates differentiation with respect to t; μ is the solar gravitational constant $\mu_0 = GM$ reduced by the constant αc representing the repulsive effect of the direct radiation pressure.

These equations of motion are very similar in form to those set up by Poynting and by Plummer, but the numerical coefficients of α, in the two places outside of μ in which this parameter occurs, differ materially from theirs.† However, the formal similarity of the equations will be taken as justification for presenting only a brief account of those aspects of the solution which have also received attention at their hands, emphasizing chiefly the points in which our results differ, and will thus enable us to develop more fully certain other aspects of the problem. Thus it follows, as in Poynting, from the second of the equations (3.5) that the angular momentum H of the particle, per unit mass, decreases in accordance with the law

$$H = r^2\dot{\theta} = h - \alpha\theta, \tag{3.6}$$

where h is its initial value. On employing this integral to eliminate the time in the radial equation, we readily find as the equation of the orbit

$$\frac{d^2u}{d\theta^2} + \frac{\alpha}{H}\frac{du}{d\theta} + u = \frac{\mu}{H^2}, \tag{3.7}$$

where u is the reciprocal $1/r$ of the radius.

Our primary problem is the solution of the equation (3.7) of the orbit, for the problem of determining the position of the particle in the orbit $u = u(\theta)$ at time t is then reduced, in principal, to the solution of (3.6) by a quadrature. Now, although we shall be but little concerned with the rigorous solution of these equations, it is nevertheless of considerable value to show that the orbital equation is reducible to a Bessel equation, for this will enable us to

* We shall in the following express our results in terms of this case of principal interest here; if it is desired to apply them to cases in which this interpretation is inapt, it is only necessary to replace this "effective" $a\rho$ by $3m/4A$.

† For purposes of comparing the results obtained here with those in the literature referred to above, we note that if we characterize the terms $-2\alpha\dot{r}/r^2$, $-\alpha\dot{\theta}/r$ in the radial and transverse accelerations in (3.5) by the number pair (2, 1), then Poynting's original equations are characterized by the pair (2/3, 2/3), those implied by Poynting in the note appended at the end, and used by Plummer in his second paper, by (4/3, 1/3), those used by Plummer in his first paper by (1/3, 1/3), and those implied by Larmor's work by (1, 1), or by (2, 2) if his aberrational effect is taken into account.

obtain the more readily a general notion of the behaviour of the exact solutions, which will in turn be of considerable assistance in judging the range in which the more manageable approximate solutions, dealt with in the sequel, are valid. This reduction is accomplished by subjecting the variables (u, θ) to the simple transformation

$$\theta = h/a - \phi, \qquad u = \mu\phi z/a^2 \tag{3.8}$$

to new variables (z, ϕ); the equation (3.7) of the orbit then becomes

$$\frac{d^2z}{d\phi^2} + \frac{1}{\phi}\frac{dz}{d\phi} + \left(1 - \frac{1}{\phi^2}\right)z = \frac{1}{\phi^3}, \tag{3.9}$$

an inhomogeneous equation for z whose complementary function is a Bessel function of order 1 and argument ϕ. The equation (3.6), determining the position of the particle at time t, may then be written in the convenient form

$$\dot{\phi} + au^2\phi = 0. \tag{3.10}$$

In the applications of greatest interest we shall find that $\phi >> 1$, which suggests the use of the well-known asymptotic expansions for the complementary function,* and of a corresponding expansion for the particular integral, to which we now turn our attention. From this standpoint the simplest particular integral—which is at the same time the most interesting for the following—is that one which does not partake of the characteristic oscillatory nature of the Bessel functions for large values of the argument ϕ; the asymptotic series for this unique integral, for values of $\phi >> 1$, is given by the divergent series

$$\phi z \sim \frac{1}{\phi^2}\left\{1 - \frac{8}{\phi^2} + \ . \ . \ . \ + \frac{(-)^n 2^{2n} n!\,(n+1)!}{\phi^{2n}} + \ . \ . \ .\right\}, \tag{3.11}$$

as may be verified by substitution in (3.9). Although of little practical application, it is of some theoretical interest to remark that *any* solution of (3.7) can be shown to assume the form

$$\phi z = P(\phi) + Q(\phi)\log\phi + \tfrac{1}{4}\phi\,(\log\phi)^2 J_1(\phi), \tag{3.12}$$

where J_1 is the Bessel coefficient of order 1 and P, Q are even functions which are analytic in the neighbourhood of the origin $\phi = 0$. The leading terms in the expansion of u, or rather of the function ϕz which only differs from it by a constant factor, are then given by

$$\phi z = -\tfrac{1}{2}\log\phi - 2A + \tfrac{1}{8}\phi\,(\log\phi)^2 + A\phi^2\log\phi + B\phi^2 + \cdots, \tag{3.13}$$

where A and B are arbitrary constants involved in the complementary function; it is to be observed that the first term, which is of most importance for sufficiently small ϕ, is common to *all* solutions of the orbital equation.

One immediate consequence which can be drawn from this last result is

* For the properties here used of the Bessel functions, see E. T. Whittaker and G. N. Watson, *Modern Analysis*, Chap. 17 (5th ed., Cambridge, 1920); the asymptotic series in question are developed in Sections 17.5, 17.6.

that, in case $\mu \neq 0$, r cannot remain indefinitely within finite non-zero limits; the particle must either fall into the Sun or escape to infinity. For if we assume that r does lie within such limits, it follows from (3.10) that $\phi \to 0$ as $t \to \infty$, and hence from the dominant term in (3.13) that $\phi z \to \infty$, which is contrary to assumption. This means, in the case $\mu > 0$, in which the direct radiation pressure is not enough to reverse the direction of effective gravity, that the particle either escapes completely from the solar system ($\phi z \to 0$) or is drawn into the Sun (ϕz sufficiently large), and in the case $\mu < 0$, in which effective gravity is reversed, that it is blown away, unless its initial circumstances of motion are such that it happens to hit the solar surface first. For completeness we note that the physically unimportant exceptional case $\mu = 0$, in which gravity is just annulled by radiation pressure, allows the further possibility that the particle is in or approaches the unstable limiting position $r =$ const., $\phi = 0$. In short, there are no finite orbits in the cases $\mu \neq 0$, for all either intersect the Sun's surface or go to infinity—and even in the exceptional case $\mu = 0$, there is in addition only a possible motion of limitation.

In the applications α/h is small, and for many purposes it suffices to retain only terms of first order in this ratio. The equation (3.7) of the orbit then reduces to

$$\frac{d^2u}{d\theta^2} + \frac{\alpha}{h}\frac{du}{d\theta} + u = \frac{\mu}{h^2}\left(1 + 2\frac{\alpha}{h}\theta\right), \tag{3.14}$$

and is readily handled by standard perturbation theory.

4. *The Particular Integral. Resisting Medium.*—The first and most striking consequence of the retardation is that stressed by Poynting—the gradual loss of angular momentum will eventually result in any particle being swept into the Sun, provided only that the solar gravitational pull is not counterbalanced by radiation pressure. It follows immediately from (3.6) that such a particle cannot make as many as $h/2\pi\alpha$ revolutions about the Sun; if we take as the initial value h that for a circular orbit of radius r, under effective gravity μ, this upper limit is given by

$$N = \frac{(\mu r)^{\frac{1}{2}}}{2\pi\alpha} = 2{\cdot}82 \times 10^7 a\rho(R\mu/\mu_0)^{\frac{1}{2}}, \tag{4.1}$$

where R is the initial distance r (in cm.) expressed in astronomical units. Thus a particle of radius 1 cm. and density 5·5, starting from a circular orbit at the Earth's distance, cannot make as many as $1{\cdot}55 \times 10^8$ revolutions, in substantial agreement with the conclusions reached by Poynting.

In order to obtain an estimate of the times involved, we consider the motion of a particle whose orbit is given by the particular integral (3.11) of the equation of the orbit. Now the magnitude of the second term in the expansion of z is greatest when ϕ is least, and the least value which ϕ can assume is that when the particle is at the surface of the Sun; this is essentially the value of h/α for a circular orbit at the distance $R = 4{\cdot}65 \times 10^{-3}$ A.U. of the Sun's surface. But here, by (4.1), the ratio $1 : \phi^2$ is only $6{\cdot}9 \times 10^{-15}\mu_0/\mu(a\rho)^2$, and for the extreme case in which the direct radiation pressure is almost sufficient to counterbalance gravity—for which the effective $a\rho \sim 5{\cdot}7 \times 10^{-5}$—the ratio is still only of the order $2{\cdot}1 \times 10^{-6}\mu_0/\mu$. Hence, unless the pressure is very close indeed to the solar gravitational pull, we may safely take as the equation of the orbit the asymptotic expansion (3.11); carrying only the first two terms of this series, we find as the equation of the spiral orbit

$$r = \alpha^2\mu^{-1}(\phi^2 + 8 + \ldots), \tag{4.2}$$

where the next term is of the order ϕ^{-2}. On expressing ϕ in terms of r, the differential equation (3.10) assumes the form

$$2\alpha dt + r(1 + 8\alpha^2/\mu r + \ldots)dr = 0, \tag{4.3}$$

and on integrating this equation we find that a particle initially at the distance r from the Sun will be at the distance r_1 at time

$$t = (r^2 - r_1^2)/4\alpha + 4\alpha(r - r_1)/\mu + \cdots \tag{4.4}$$

Even for cases in which radiation pressure is a respectable fraction of gravitation—say as much as 99 per cent.—the first term in (4.4) gives a remarkably good approximation to the time it takes a particle to fall from a distance r to the solar surface at r_1. If in addition r is large compared with the Sun's radius, we may further simplify this to the form

$$t = r^2/4\alpha \text{ seconds} = 7{\cdot}0 \times 10^6 a\rho R^2 \text{ years}, \tag{4.5}$$

where R is the initial distance r expressed in astronomical units. An illuminating alternative form of this result is obtained by expressing the time of fall t in terms of the period P of the initial, essentially circular, orbit described by the particle at distance r under the effective μ; we may then write

$$t = \tfrac{1}{4}NP, \quad \text{where} \quad P = 2\pi\mu^{-\frac{1}{2}}r^{\frac{3}{2}}, \tag{4.6}$$

and N is the number of revolutions (4.1) in which the particle falls into the Sun.

To give some numerical results, we note that it follows from (4.5) that a particle of radius 1 cm. and density 5·5 (that of the Earth), starting from the distance of the Earth, will be swept into the Sun in $3{\cdot}9 \times 10^7$ years, while a similar particle of radius 10^{-3} cm. can survive but $3{\cdot}9 \times 10^4$ years. The lower limit for the time of fall of a particle from the distance of the Earth is of the order of 400 years, and is realized by a particle whose effective $a\rho$ is near, but not too close to, the critical value $5{\cdot}7 \times 10^{-5}$ at which radiation pressure just annuls the solar gravitation. Alternatively, we may express the result above in terms of the size of the region which is swept clear of the particles initially within it in some given period of time t. Thus particles of a *given* $a\rho$ are swept out of a region of radius $11{\cdot}9\ (a\rho)^{-\frac{1}{2}}$ A.U. in 10^9 years; the upper limit of this expression, as the effective $a\rho$ approaches the critical value, is some 1600 A.U., and represents the dimensions of the greatest region in which all particles of *some* effective $a\rho$ (near the critical value) can be swept into the Sun in this time 10^9 years.

It is clear that this effect cannot be ignored in cosmogonic speculations employing a resisting medium, and that it may be expected to rule out certain forms of such a medium. The application to a given type of medium depends sharply on the nature of the particles constituting it, and turns in the last analysis upon the evaluation of the direct radiation pressure f, for this determines the rate cf at which radiation is absorbed and re-emitted or scattered, and therefore the rate at which angular momentum is lost. The same remarks apply to questions of the origin, nature and continued existence of the medium responsible for the zodiacal light and the counterglow, questions touched upon by Poynting in several of his stimulating addresses on the astronomical consequences of the pressure of light. On the other hand, this effect can hardly be of appreciable influence on the galactic scale, for the time of fall (4.5)—which, it is to be noted, does not obey Kepler's third law—increases so rapidly with distance that no observable effects on interstellar matter are to be expected. This remark applies in particular to bodies describing orbits around the galaxy as a whole (in spite of the fact that the parameter α may then be some 10^{11} times its value for the Sun), and to the sweeping-out of the bulk of the matter constituting the enormous diffuse nebulæ associated with certain stars.

In all of the above we have been concerned with particles whose effective $a\rho$ was not too close to the critical value, defined implicitly by $ac = \mu_0$, at which the direction of effective gravity is reversed—say, no nearer than 1 per cent. for good measure. Although we have with this surely covered all cases of possible astronomical importance, it may nevertheless be not without interest to complete the analysis by adding a few words concerning the behaviour of a particle whose effective $a\rho$ is so small that the asymptotic expansion breaks down. Retaining only the first two terms in the expansion (3.13)—the only ones which do not vanish as $\phi \rightarrow 0$—the orbit is the spiral defined by

$$u = -\tfrac{1}{2}\mu\alpha^{-2} \log \phi + \text{const.}, \tag{4.7}$$

and on integrating (3.10) we find as the time taken by such a particle in falling from a distance r to r_1

$$t = 2\alpha(r - r_1)/\mu. \tag{4.8}$$

Hence in the μ-range here under consideration, a further decrease causes the time of fall to increase—toward the value ∞ for $\mu \rightarrow 0$, representing in the limit a particle at rest under the Sun's attraction and an equal and opposite radiation pressure. This particular integral u of course vanishes for effective gravity $\mu = 0$, and cannot be used for $\mu < 0$ in the regions considered above, without the addition of a complementary function, for it is there negative.

5. *The Complete Integral. Encke's Comet.*—We conclude with a brief account of the first-order perturbations caused by radiation drag, as obtained by the usual methods from the approximate equation (3.14) of the orbit.* In much of this Section we limit ourselves essentially to a statement of the results, as their derivation follows very closely that of H. C. Plummer † based on the similar perturbing force obtained by Poynting.

* The more accurate long-time treatment, analogous to the development of the previous Section 4 in that it is based on the leading terms in the asymptotic expansion of the solution of the Bessel equation (3.9), becomes here unwieldy—and of course reduces to the present treatment on retaining only terms of first order in a/h.

† *M.N.*, **65**, 229, 1905 ; **67**, 63, 1906.

To this order, then, the equation of the orbit is readily shown to be given by

$$u = p^{-1}\left\{1 + 2\frac{\alpha}{h}\theta + e\left(1 - \frac{\alpha}{2h}\theta\right)\cos(\theta - \gamma)\right\}, \tag{5.1}$$

where $p = h^2/\mu$. Consider first the more important case in which the osculating orbit is an ellipse. The first order secular perturbations, during one revolution, of the elements of this osculating ellipse are then found to be

$$\frac{\Delta p}{p} = -\frac{4\pi\alpha}{h}, \qquad \frac{\Delta e}{e} = -\frac{5\pi\alpha}{h}, \qquad \Delta\gamma = 0,$$
$$\frac{\Delta n}{n} = -\frac{3}{2}\frac{\Delta a}{a} = \frac{2 + 3e^2}{1 - e^2}\frac{3\pi\alpha}{h}. \tag{5.2}$$

From the first of these equations it follows that the semi-latus rectum of the ellipse diminishes at the same rate as the distance in the spiral orbit discussed in Section 4 above, and from the second that the eccentricity decreases at an even faster specific rate; not only does the orbit of such a particle spiral in toward the Sun, but it at the same time becomes more circular. The principal modification caused by the inclusion of general relativity, for times short compared with those in which the Schwarzschild metric is appreciably altered because of the radiation of mass by the Sun, is the perihelion advance

$$\frac{\Delta\gamma}{2\pi} = \frac{3\mu^2}{c^2 h^2}; \tag{5.3}$$

although this effect is quite comparable in magnitude with those here considered, it would seem to be of little cosmogonic significance.

In the papers referred to above, Plummer has examined the possibility of explaining the anomalous acceleration of the mean motion of Encke's comet, which exhibits itself in a decrease of its period, of about $3\frac{1}{2}$ years, by some $2\frac{1}{2}$ hours on each return. He found, in his second note, that the anomaly could be explained by assuming that the comet's head consisted of particles of the order of 2×10^{-3} cm. radius, but that the effective gravitation would then be reduced, because of the radiation pressure, by one part in 125, and this would lead to an untenable decrease (of the order of one part in 600) of the assumed mean distance. The situation is not much improved by the present revision of Poynting's expressions for the drag; on employing Abbot's value for the solar constant and Backlund's elements for the orbit, as quoted by Plummer, the equations (5.2) yield

$$a\rho = 0{\cdot}016, \qquad \alpha c/\mu_0 = 0{\cdot}0036. \tag{5.4}$$

Thus the particles, of assumed density 5·5, can be some half again as large as in Plummer's work (or twice as large, if we reduce his to the size they would have under the value of the solar constant here used), and the decrease in effective gravity is now of the order of only one part in 280. But this is presumably still inadmissibly large—if we throw all the error on to the assumed mean distance, for example, it would lead to a discrepancy in the parameter of the orbit of the order of one part in 10^4.

For cases in which the direct radiation pressure is so enhanced by diffraction or resonance that it reverses the direction of effective gravity, as in the case of the material constituting the tails of comets, the retardation is correspondingly increased. But here there is but little time for the production of observable effects, for, unless other forces are at work, the particle is lost to the solar system in a relatively insignificant time. If we take the effective μ to represent a repulsion N times as strong as gravity, then

$$\alpha = (N+1)\mu_0/c = (N+1)4 \cdot 41 \times 10^{15}, \tag{5.5}$$

and the equation of the orbit is

$$u = N\mu_0 h^{-2}\left\{e\left(1 - \frac{\alpha}{2h}\theta\right)\cos(\theta-\gamma) - \left(1 + 2\frac{\alpha}{h}\theta\right)\right\}. \tag{5.6}$$

Assume that at perihelion the particle has ν times the circular velocity at that distance ; the constant e, which is essentially the eccentricity of the tangent hyperbola described under the repulsive force $N\mu_0/r^2$ from the same initial conditions, is then found to be

$$e = 1 + \nu^2/N. \tag{5.7}$$

For cases in which this differs but little from unity, the order of the effect is illustrated by the fact that it results in a small decrease η of the semi-angle between asymptotes of amount

$$\eta \sim \frac{\nu^2}{3N}\frac{\alpha}{h}. \tag{5.8}$$

In all cases there is a tendency for the particle to lose energy, as well as angular momentum, to the radiation field—where it is interpretable as Doppler shifts. The mechanical aspects of this may perhaps be most clearly brought out by returning to the equations of motion (3.5) and computing the rate of change of the sum

$$E = T + U = \tfrac{1}{2}(\dot{r}^2 + r^2\dot{\theta}^2) - \frac{\mu_0}{r} \tag{5.9}$$

of the kinetic energy T and the gravitational energy U, per unit mass of the particle. The result of this computation is that

$$\frac{dE}{dt} = -\frac{dV}{dt} - \frac{\alpha}{r^2}(2\dot{r}^2 + r^2\dot{\theta}^2), \quad \text{where} \quad V = -\frac{\alpha c}{r}, \tag{5.10}$$

may be thought of an additional potential energy attributable to the direct radiation pressure, and the second, essentially negative term is that due to the non-conservative dragging forces. It therefore follows that the sum $T+U+V$ must decrease monotonically ; but then on describing a closed circuit, or an arc beginning and ending at the same distance r from the Sun, the kinetic energy T must alone suffer the entire loss, since U and V return to their original values. Thus, in particular, a particle entering the solar system from without must leave with a smaller limiting velocity—if, indeed, it ever gets out again, for unless its entering velocity is sufficiently in excess of the parabolic velocity under effective gravity μ, it will be captured by the Sun.

6. *Summary.*—This examination, from the standpoint of the theory of relativity, of the problem of the retardation of motion of a body receiving and re-emitting radiation, has established the existence of a drag of the same general nature as that predicted by Poynting on classical grounds. The actual expressions for the retarding force, for the case of a spherical particle reflecting or absorbing and re-emitting the radiation it receives from the Sun, differ from those obtained by Poynting and by Larmor, even in the approximation to which their considerations were restricted, and it was therefore considered desirable to supplement the rigorous derivation by a brief account of the specific effects arising from the drag in the justifiable approximation in which only terms of first order in the velocity are retained. Those effects of greatest astronomical interest would seem to be the secular decrease of the mean distance of the particle from the Sun—of roughly three times the amount predicted by Poynting and by Plummer, or one-half of that implied by the later considerations of Larmor—and the similar decrease in the eccentricity of the osculating orbit, all on the supposition that the direct radiation pressure is not strong enough to reverse the direction of effective gravity. The point of most significance for cosmogony is that this drag, attendant on the pressure due to the solar radiation, constitutes a force which is effective in clearing the neighbourhood of the Sun of small particles in astronomically significant times.

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TWO VISUAL BINARY ORBITS.

R. v. d. R. Woolley and L. S. T. Symms.

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This note describes the calculation of the orbits of two visual binary stars which have many features in common. Both orbits are highly inclined and highly eccentric, and in both cases observations in at least one quadrant are lacking, and the important part of the orbit round periastron is unobserved. On the other hand, both stars have completed at least one revolution, so that some elements, notably the period, can be determined with accuracy. Lastly, both stars are approaching periastron, and it is just possible that a careful examination of the spectrum will show doubled lines, as velocity differences of about 20 km./sec. may be expected. Needless to say, if spectroscopic observations of velocity difference are secured they will not only yield excellent data for improving the orbital elements at their weak points, but will also give values for the parallaxes and other important quantities connected with the stars.

In both cases the new orbits have been computed by applying differential corrections to published orbits. The elements of the orbit may be expressed in the form A, B, F, G, e, T, n, such that the rectangular co-ordinates (x, y) of the companion referred to the primary as origin are given by