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Kinematics of an Ultraelastic Rough Ball

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A rough ball which conserves kinetic energy exhibits unexpected behavior after a single bounce and bizarre behavior after three bounces against parallel surfaces. The Wham-O *Super-Ball*® (registered by Wham-O Corporation, 835 E. El Monte St., San Gabriel, Calif. 91776), appears to approximate this behavior and provides an inexpensive and readily available model of kinematics quite different from that of a point mass or smooth ball. The analysis is most strikingly illustrated by the fact that the ball returns to the hand after three collisions with the floor, the underside of a table, and the floor. Some questions are raised concerning the dynamics of the collision.

INTRODUCTION

"Angle of incidence equals angle of reflection" is a commonly quoted but, of course, not universal result, and it is useful to demonstrate to a freshman or high school physics class that real objects exhibit behavior equally predictable but in some cases quite different. In particular, a perfectly rough ball which conserves kinetic energy behaves in such an unexpected way that it is difficult to pick up after it has bounced twice upon the floor, and, more bizarre, it returns to the hand on being thrown to the floor in such a way that it bounces from the underside of a table as in Fig. 1.[1] It turns out that the two

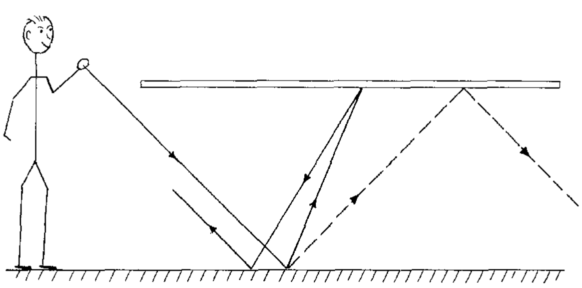

FIG. 1. A *Super Ball* seems to return to the hand after bouncing against the underside of a table, while the expectation is for it to continue bouncing between the floor and the table as shown by the dotted line.

assumptions of conservation of kinetic energy and no slip at the contacting surfaces predict qualitatively the observed behavior of the *Super Ball*. It is not at first clear, however, how even a microscopically perfectly elastic body can conserve kinetic energy in such collisions, and this is discussed later.

Two primary assumptions suffice to determine the trajectories of the *Super Ball*: (a) Kinetic energy is conserved during a collision (the rotational plus the translational energy of the ball is the same after collision as before). (b) There is no slip at the point of contact (the ball is "perfectly rough"). Two further conditions follow from the laws of mechanics: (c) Angular momentum L about the point of contact is conserved during the collision. (d) The normal component of velocity is reversed by a collision. (c) follows from the approximation that the contact occurs at a point and that all forces act through that point, so their moment about that point is zero. (d) follows from the assumption of linear equations of motion and the observation that, in the special case of normal incidence without spin, all kinetic energy after the collision must again be in the normal velocity, which is therefore preserved with change of sign, as a consequence of (a). Since the spin, the normal velocity and the tangential velocity after collision are all assumed to be linear functions of the velocities before the collision, the cross coupling between normal velocity and the other two degrees of freedom is shown by this special case to be zero under *all* initial conditions.

Figure 2 shows the elementary collision against a hard surface oriented in the xz plane. Before collision, the body has a velocity V_b along x and an angular velocity ω_b along z. After the bounce, these variables become V_a and ω_a, respectively.

[1] This was first demonstrated to me by L. W. Alvarez with a Wham-O *Super Ball*.

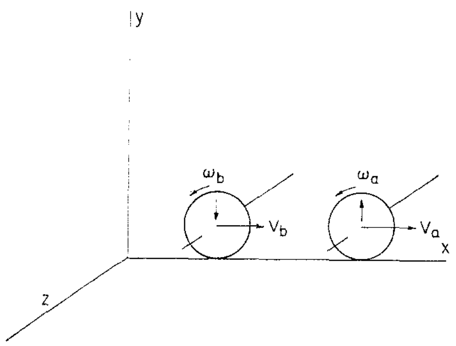

FIG. 2. The *Super Ball* has velocity V_b and spin ω_b before bouncing from the xz plane. After the bounce the velocity and spin are V_a and ω_a respectively. The normal velocity along y is simply reversed in the collision.

The subscriptions b and a refer to conditions *before* or *after* a given collision.

I. COLLISION KINETICS

Let the moment of inertia about the z axis be $I\equiv\alpha MR^2$, with $\alpha=\frac{2}{5}$ for a uniform sphere. M is the mass, and R the radius of the ball. Then the kinetic energy, including only the spin ω and the x component of the velocity V, is

$$K=\tfrac{1}{2}MV^2+\tfrac{1}{2}I\omega^2=M/2(V^2+\alpha\omega^2R^2)$$
$$=M/2(V^2+\alpha C^2), \qquad (1)$$

where $C=\omega R$ has been introduced to simplify the equations but in fact represents the (signed) peripheral velocity of the ball's surface adjacent to the wall, relative to the center.

The angular momentum about the point of contact is then (for the ball close to the wall)

$$L=I\omega-MRV=(\alpha MR^2\omega-MRV)$$
$$=MR(\alpha C-V), \qquad (2)$$

in which L is expressed as the angular momentum about the center of the ball plus that of the center of mass about the point of contact.

Thus, equating the kinetic energy and the angular momentum before and after a bounce, we have

$$K_b=M/2(V_b^2+\alpha C_b^2)=M/2(V_a^2+\alpha C_a^2)=K_a, \qquad (3)$$

and

$$L_b=MR(\alpha C_b-V_b)=MR(\alpha C_a-V_a)=L_a, \qquad (4)$$

in which the velocities *before* are V_b, C_b and those *after* are V_a, C_a. Equations (3) and (4) become

$$V_b^2+\alpha C_b^2=V_a^2+\alpha C_a^2, \qquad (5)$$

and

$$V_b-\alpha C_b=V_a-\alpha C_a; \qquad (6)$$

and simplification gives

$$\alpha(C_b-C_a)(C_b+C_a)=(V_a-V_b)(V_a+V_b), \qquad (5')$$

$$\alpha(C_b-C_a)=-(V_a-V_b). \qquad (6')$$

Dividing, we obtain an interesting result,

$$C_b+C_a=-(V_a+V_b),$$

or

$$V_b+C_b=-(V_a+C_a), \qquad (7)$$

showing that the parallel velocity of the contacting portion of the ball, $S\equiv V+C$, is precisely *reversed* by a bounce.

Subtracting Eq. (7) from Eq. (6) we have

$$(\alpha-1)C_b-(\alpha+1)C_a=2V_b,$$

or

$$C_a=[(\alpha-1)/(\alpha+1)]C_b-[2/(\alpha+1)]V_b; \qquad (8)$$

and resubstituting in Eq. (7) we find

$$V_a=-[2\alpha/(\alpha+1)]C_b-[(\alpha-1)/(\alpha+1)]V_b. \qquad (9)$$

Equations (8) and (9) thus give the spin and velocity after collision.

For example, if a spherical ball approaches the wall with $\omega=0$, $V_b=V_\perp$, i.e. with no spin and at an angle $\theta_b=45°$ with respect to the normal, then $C_b=0$ and

$$C_a=(-10/7)V_\perp, \qquad V_a=\tfrac{3}{7}V_\perp,$$

so that the ball on bouncing makes an angle $\theta_a=\tan^{-1}(3/7)=23.2°$ with the outward normal from the wall. [See Fig. 3(a).] Similarly, a ball

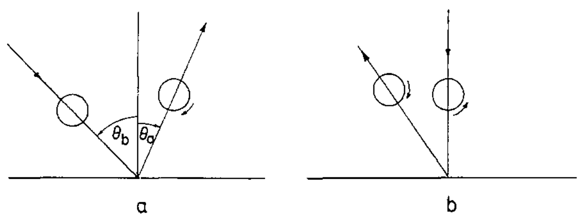

FIG. 3. (a) A *Super Ball* with zero spin bounces from a wall with $\tan\theta_a = 3/7 \tan\theta_b$ and with spin velocity 10/7 of its initial horizontal velocity. (b) A ball with initial spin has its spin velocity reduced to 3/7 and reversed, and acquires transverse velocity as shown.

with initial spin velocity C_b but with no translational velocity V_b leaves the wall with

$$C_a = -\tfrac{3}{7}C_b, \qquad V_a = -\tfrac{4}{7}C_b$$

as indicated in Fig. 3b.

II. TRAJECTORIES

Equations (8) and (9), and assumption (d) suffice to allow the path of a *Super Ball* to be traced through any number of collisions with rigid walls. One simply uses the geometry to determine V_b and $V_\perp$ before any new collision, from the V_a and $V_\perp$ after the previous one. One then applies Eqs. (8) and (9), and assumption (d) and reprojects the velocity to obtain the initial velocities for the next collision.

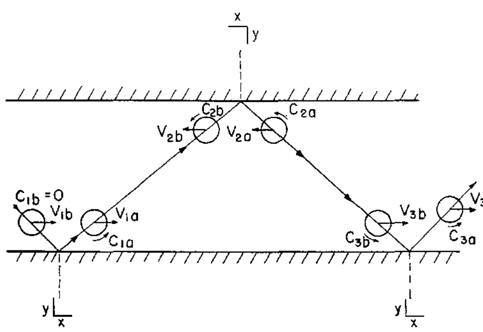

FIG. 4. A *Super Ball* makes three successive bounces between two parallel walls. The positive sense of velocity and spin is shown before and after each of the three collisions.

If we consider throwing the ball to the floor and allowing it to bounce against the underside of a table before bouncing again from the floor, we have the situation of Fig. 4, with the coordinate system of Fig. 1, except that for bounce 2, the coordinates used are shown in the inset, in order to employ the standard Eqs. (8) and (9). For a ball without initial spin ($C_{1b}=0$), repeated application of these equations gives V_{1a}, C_{1a}; V_{2b}, C_{2b}; V_{2a}, C_{2a}; V_{3b}, C_{3b}; and finally V_{3a}, C_{3a}.

$$V_{1a} = -[(\alpha-1)/(\alpha+1)]V,$$

$$C_{1a} = [-2/(\alpha+1)]V,$$

$$V_{2b} = [(\alpha-1)/(\alpha+1)]V,$$

$$V_{2a} = -\left(\frac{2\alpha}{\alpha+1}\right)\left(\frac{-2}{\alpha+1}\right)V - \left(\frac{\alpha-1}{\alpha+1}\right)^2 V$$

$$= [V/(\alpha+1)^2][4\alpha-(\alpha-1)^2],$$

$$C_{2a} = \left(\frac{\alpha-1}{\alpha+1}\right)\left(\frac{-2}{\alpha+1}\right)V - \left(\frac{2}{\alpha+1}\right)\left(\frac{\alpha-1}{\alpha+1}\right)V$$

$$= [V/(\alpha+1)^2][-2(\alpha-1)-2(\alpha-1)],$$

$$V_{3b} = [V/(\alpha+1)^2][(\alpha-1)^2-4\alpha],$$

$$C_{3b} = C_{2a}.$$

$$V_{3a} = [V/(\alpha+1)^3][-\alpha^3+15\alpha^2-15\alpha+1]. \qquad (10a)$$

$$C_{3a} = [V/(\alpha+1)^3][-6\alpha^2+20\alpha-6]. \qquad (10b)$$

For a sphere $\alpha = \frac{2}{5}$, and

$$V_{3a} = -\frac{333}{343}V, \qquad C_{3a} = \frac{130}{343}V.$$

Thus a sphere *returns* after three bounces with but 3% lower velocity than it had when it started. An accurate reproduction of its trajectory is presented in Fig. 5. It is worth noting[2] that a

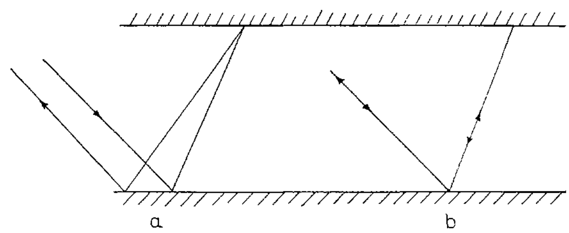

FIG. 5. A *Super Ball* thrown without spin will follow the path indicated in 5(a) in bouncing from the floor to the underside of a table and back to the floor. The tangent of the angle of bounce is 3% greater than that of the angle of incidence. For comparison, the trajectory of a body with $I = \frac{1}{3}MR^2$ is shown in 5(b)—it returns precisely along its initial path.

[2] R. Friedberg, private communication, December 1967.

body with $\alpha=\frac{1}{3}$ (e.g., a ball containing a central high-density mass $\frac{1}{5}$ that of the original ball) returns without spin and parallel to the original direction. Although gravity has been neglected in this calculation, its inclusion changes nothing so long as the ball does in fact strike the underside of the table.

It is difficult to make all the required substitutions accurately, and advantage may be taken of the linear form of Eqs. (8) and (9) to define a collision matrix M such that the velocity after a collision $\mathbf{V}_a$ is related to the velocity before, $\mathbf{V}_b$, by

$$\mathbf{V}_a=\mathbf{M}\cdot\mathbf{V}_b, \tag{11}$$

in which $\mathbf{V}_a$ and $\mathbf{V}_b$ are row vectors with three elements representing the parallel velocity V, the peripheral velocity C, and the normal velocity V_n, respectively. The collision matrix is then

$$\mathbf{M}=\begin{bmatrix}(1-\alpha)/(\alpha+1) & -2\alpha/(\alpha+1) & 0\\ -2/(\alpha+1) & (\alpha-1)/(\alpha+1) & 0\\ 0 & 0 & -1\end{bmatrix}$$

$$=-1/(\alpha+1)\begin{bmatrix}\alpha-1 & 2\alpha & 0\\ 2 & 1-\alpha & 0\\ 0 & 0 & \alpha+1\end{bmatrix}. \tag{12}$$

Equation (12) has been used in a computer program to provide the data for Fig. 6, which shows the trajectory of a *Super Ball* and of a smooth ball within a unit square.[3] In computing the trajectories of Fig. 6, before applying the collision matrix, the computer after each collision determines the point of impact of the next collision and transforms the components of the velocity vector to a new wall-oriented frame of reference. This velocity transformation is done by matrix multiplication by matrices $\mathbf{D}_1$, $\mathbf{D}_2$, $\mathbf{D}_3$, according as the next side struck is at angle $\pi/2$, π, or $3\pi/2$ with respect to the previous side of the square.

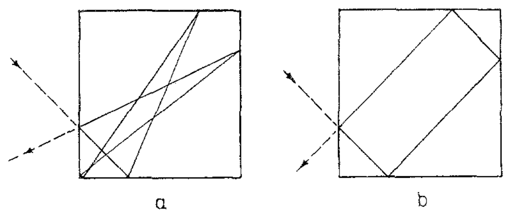

Fig. 6. (a) Shows the path of an ultraelastic rough ball. (b) Shows the path of a smooth one inside a rigid square.

Following Romer[4] we could have obtained Eqs. (10) by repeated matrix multiplication: $\mathbf{V}_{1a}=\mathbf{M}\cdot\mathbf{V}_{1b}$; $\mathbf{V}_{2b}=\mathbf{D}_2\cdot\mathbf{V}_{1a}$; $\mathbf{V}_{2a}=\mathbf{M}\cdot\mathbf{V}_{2b}$; $\mathbf{V}_{3b}=\mathbf{D}_2\cdot\mathbf{V}_{2a}$; $\mathbf{V}_{3a}=\mathbf{M}\cdot\mathbf{V}_{3b}$; or $\mathbf{V}_{3a}=\mathbf{M}\mathbf{D}_2\mathbf{M}\mathbf{D}_2\mathbf{M}\mathbf{V}_{1b}$. Here

$$\mathbf{D}_2=\begin{bmatrix}-1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & -1\end{bmatrix}.$$

$\mathbf{D}_2$ does not commute with $\mathbf{M}$, the product of the five matrices, giving

$$\mathbf{V}_{3a}=1/(\alpha+1)^3=\begin{bmatrix}-\alpha^3+15\alpha^2-15\alpha+1 & -6\alpha^3+20\alpha^2-6\alpha & 0\\ -6\alpha^2+20\alpha-6 & \alpha^3-15\alpha^2+15\alpha-1 & 0\\ 0 & 0 & -(\alpha+1)^3\end{bmatrix},$$

which reduce to Eqs. (10) for the special case $C_{1b}=0$.

[3] The program was written in APL\360 with the help of T. M. Garwin. A copy of the program and of the output for the cases in Fig. 6 is available to anyone inquiring.

[4] R. H. Romer, Amer. J. Phys. **35**, 862 (1967). I am indebted to the referee, who called to my attention this paper. My purpose is to explain the behavior of the *Super Ball*, but Romer's elegant matrix treatment of binary one-dimensional collisions could be applied directly to the exchange of energy between the two internal degrees of freedom (velocity and spin) by a simple transformation. I record the algebra leading to Eqs. (10) to motivate the use of the matrices **M** and **D**.

III. DISCUSSION

In going from Eqs. (4) and (6) to Eq. (7), we have tacitly excluded one solution, i.e. $C_a = C_b$, $V_a = V_b$. In addition, we have nowhere overtly used assumption (b) that there is no slip at the point of contact. In fact, conservation of energy [assumption (a)] as well as conditions (c) and (d) are consistent with this excluded solution, which would indeed be predicted if there were a frictionless slip between ball and wall. The question is more complicated, however, as is apparent from the result of Eq. (6), namely that the parallel velocity of the point of contact is just *reversed* by the chosen rough energy-conserving collision. Clearly, if the rough ball were again brought in contact with the wall immediately after such an energy-conserving collision and were held there for a similar time, the parallel surface velocity would once again reverse in this second collision, and the ball would rebound just as if it had been perfectly smooth and slipped freely.[5] Equally clearly, if the period of contact were insufficient for the shear wave to penetrate through the ball, the ball would be left in a state of internal excitation (torsional vibration) which would contradict assumption (a), which assumes that no energy is present in modes other than those of uniform translation and of spin. Thus, it seems likely that the striking behavior of the *Super Ball* is due not so much to an extremely low internal dissipation for elastic waves as it is to a near equality between periods for a bounce of the ball from the wall and for a torsional oscillation of the ball. It is this further requirement on the relative speeds of compression and shear waves which leads me to use the term "ultraelastic" in this paper. Even for zero internal friction, the bounce can never really be perfectly elastic because the higher modes of vibration are not harmonically related to the fundamental, and so it retains some excitation after the bounce.

The conceptual model of a collision thus has the normal velocity of the ball, upon contact, undergoing a half-period of oscillation and thus reversing. At the same time, the surface of the ball in contact with the wall comes abruptly to rest and a shear wave propagates into the ball, and, reflecting from the free surface, reverses the initial velocity of the surface near the wall. The process is easy to imagine for a rough elastic slab colliding with a wall, or for a spinning sphere or disk, mounted on a fixed axis, the surface of which is suddenly brought to rest. It is simpler to think about a ring rather than a sphere or other solid body. For a ring, $\alpha = 1$, and Eq. (12) shows that such a body transforms *all* its energy of translation into spin (and vice versa) at each collision.

Although the problem is too complex to be treated here, it would be of some interest to consider the dynamics of a collision in more quantitative detail, probably taking into account the complete modal spectrum, in order to understand ultimate limits on elasticity of a collision.

[5] Indeed, $\mathbf{M}^2 = \mathbf{I}$, so that two bounces on the floor should restore the initial parallel velocity and spin. I do not know to what precision a real *Super Ball* recovers its initial spin on the second bounce if, e.g., the initial parallel velocity is zero.

ACKNOWLEDGMENTS

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